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On Spatially Homogeneous Solutions of a Modified Boltzmann Equation for Fermi–Dirac Particles

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The paper considers a modified spatially homogeneous Boltzmann equation for Fermi–Dirac particles (BFD). We prove that for the BFD equation there are only two classes of equilibria: the first ones are Fermi–Dirac distributions, the second ones are characteristic functions of the Euclidean balls, and they can be simply classified in terms of temperatures: $T > \frac{2}{5} T_F$ and $T = \frac{2}{5} T_F$, where T_F denotes the Fermi temperature. In general we show that the L^∞-bound $0 \leqslant f \leqslant 1/\varepsilon$ derived from the equation for solutions implies the temperature inequality $T \geqslant \frac{2}{5} T_F$, and if $T > \frac{2}{5} T_F$, then f trend towards Fermi–Dirac distributions; if $T = \frac{2}{5} T_F$, then f are the second equilibria. In order to study the long-time behavior, we also prove the conservation of energy and the entropy identity, and establish the moment production estimates for hard potentials.

KEY WORDS: Boltzmann equation for Fermi–Dirac particles; moment production estimate; entropy; classification of equilibria; temperature inequality.

1. INTRODUCTION

Quantum modifications of the Boltzmann equation for Fermi–Dirac particles and for Bose–Eintein particles had been given sixty years ago[(7)] in order to study time-evolution of gases of the particles. Because of taking the quantum effects into account, the modified Boltzmann equations possess strong nonlinear structures that particularly make the investigation of long-time behavior of solutions more difficult.[(9, 12, 14)] Results obtained so far are rather incomplete even for spatially homogeneous equations.

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In this paper we study the spatially homogeneous Boltzmann equation modified for Fermi–Dirac particles. According to ref. 7, the equation is given by

$$\frac{\partial}{\partial t} f(v,t) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)[f'f'_*(1-\varepsilon f)(1-\varepsilon f_*) \\ - f f_*(1-\varepsilon f')(1-\varepsilon f'_*)]\, d\omega\, dv_*, \tag{BFD}$$

where $\varepsilon = (\frac{h}{m})^3/g$, h is the Planck's constant, m and g are the mass and the "statistical weight" of a particle. The solutions f are velocity distribution functions (or the particle number densities). The right-hand side of Eq. (BFD) is the so-called collision integral, which describes the rate of change of f due to a binary collision. The function $B(z,\omega)$ is the collision kernel which is a nonnegative Borel function of $|z|$, $|\langle z,\omega\rangle|$ only. In this paper the kernel is mainly taken for the inverse power potentials (with angular cut-off) and for the hard sphere model, i.e., the kernel B is given by[4]

$$B(z,\omega) = b(\theta)\,|z|^{\beta}, \qquad -3<\beta\leqslant 1 \tag{1.1}$$

where $\theta = \arccos(|\langle z,\omega\rangle|/|z|)$, $b(\theta)$ is strictly positive in the interval $(0,\pi/2)$ and satisfies the angular-cutoff assumption:

$$A_0 := 4\pi\int_0^{\pi/2} \sin(\theta)\, b(\theta)\, d\theta < \infty. \tag{1.2}$$

The exponent β is determined by potentials of intermolecular forces, i.e., the soft potentials $(-3<\beta<0)$, the Maxwell model $(\beta=0)$ and the hard potentials $(0<\beta\leqslant 1$, including the hard sphere model: $\beta=1$, $b(\theta)=$ const. $\cos\theta)$. Notations f_*, f' and f'_* are abbreviations of the same function f in different velocity variables, i.e., $f=f(v,\cdot)$, $f_*=f(v_*,\cdot)$, $f'=f(v',\cdot)$, $f'_*=f(v'_*,\cdot)$, where v, v_* and v', v'_* are velocities of two particles before and after their collisions respectively, and they have the following relations which are frequently used in the change of integral variables:

$$v' = v - \langle v-v_*,\omega\rangle\,\omega, \qquad v'_* = v_* + \langle v-v_*,\omega\rangle\,\omega,\ \omega\in\mathbf{S}^2$$
$$v'+v'_* = v+v_*, \qquad |v'|^2+|v'_*|^2 = |v|^2+|v_*|^2,$$
$$|\langle v'-v'_*,\omega\rangle| = |\langle v-v_*,\omega\rangle|, \qquad |v'-v'_*| = |v-v_*|.$$

In Eq. (BFD), the sign of the factor $1-\varepsilon f$ is the most important: A statistical description for the BFD model given in ref. 7 (based on the Pauli exclusion principle) implies that the factor $1-\varepsilon f$, as a ratio, should be nonnegative. This implies that solutions of Eq. (BFD) should be bounded: $0 \leqslant f \leqslant 1/\varepsilon$ on $\mathbf{R}^3 \times [0, \infty)$.

As usual, we introduce the subclasses of $L^1(\mathbf{R}^3)$:

$$L^1_s(\mathbf{R}^3) = \left\{ f \;\middle|\; \|f\|_{L^1_s} \equiv \int_{\mathbf{R}^3} |f(v)|(1+|v|^2)^{s/2}\, dv < \infty \right\}, \qquad s \geqslant 0$$

and denote $\|f\|_{L^1} = \|f\|_{L^1_0}$. Here f are real or complex valued measurable functions.

Let $Q(f)(v, t) := Q(f(\cdot, t))(v)$ be the collision integral in Eq. (BFD), i.e.,

$$Q(f)(v) = Q^+(f)(v) - Q^-(f)(v),$$

$$Q^+(f)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f' f'_*(1 - \varepsilon f)(1 - \varepsilon f_*)\, d\omega\, dv_*,$$

$$Q^-(f)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f f_*(1 - \varepsilon f')(1 - \varepsilon f'_*)\, d\omega\, dv_*.$$

It is easy to see that if the kernel $B(z, \omega)$ is given (or bounded from above) by (1.1) with (1.2), then $Q^{\pm}(f) \in L^\infty_{\mathrm{loc}}([0, \infty); L^1_1(\mathbf{R}^3))$ for all $f \in L^\infty_{\mathrm{loc}}([0, \infty); L^1_2(\mathbf{R}^3))$ satisfying $0 \leqslant f \leqslant 1/\varepsilon$.

Solutions of Eq. (BFD). Suppose the kernel B is given (or bounded from above) by (1.1) with (1.2). Given an initial datum $f_0 \in L^1_2(\mathbf{R}^3)$ satisfying $0 \leqslant f_0 \leqslant 1/\varepsilon$. We say that a function f is a mild solution of Eq. (BFD) on $\mathbf{R}^3 \times [0, \infty)$ with $f|_{t=0} = f_0$ if f is measurable in both variables $(v, t) \in \mathbf{R}^3 \times [0, \infty)$ and satisfies the following (i), (ii):

(i) $f \in L^\infty_{\mathrm{loc}}([0, \infty); L^1_2(\mathbf{R}^3))$ and $0 \leqslant f \leqslant 1/\varepsilon$ on $\mathbf{R}^3 \times [0, \infty)$.

(ii) There is a null set $Z \subset \mathbf{R}^3$ such that for all $v \in \mathbf{R}^3 \backslash Z$ and all $t \in [0, \infty)$

$$f(v, t) = f_0(v) + \int_0^t Q(f)(v, \tau)\, d\tau.$$

Applying Fubini's theorem, it is easily shown that if, instead of (ii), f satisfies

$$f(v, t) = f_0(v) + \int_0^t Q(f)(v, \tau)\, d\tau, \qquad t \in [0, \infty), \quad v \in \mathbf{R}^3 \backslash Z_t, \quad \mathrm{mes}(Z_t) = 0,$$

then f can be modified on v-null sets such that the modification of f satisfies (ii). In this sense, we do not distinguish between f and its modifications on v-null sets. In this paper, a function f is said to be a solution of Eq. (BFD) always means that f is a mild solution of Eq. (BFD).

A solution will be briefly called a conservative solution if it conserves the mass, momentum and energy, i.e., the equalities of the five moments

$$\int_{\mathbf{R}^3} f(v,t)\,\psi(v)\,dv = \int_{\mathbf{R}^3} f_0(v)\,\psi(v)\,dv, \qquad \psi(v) = 1, v_1, v_2, v_3, |v|^2$$

hold for all $t \in [0,\infty)$. Here v_i are components of v. It is easily seen that for any solution f of Eq. (BFD), we have $Q^{\pm}(f) \in L^{\infty}_{\rm loc}([0,\infty); L^1_1(\mathbf{R}^3))$ which implies that f always conserves the mass and momentum.

Entropy used in this paper for the BFD model is taken as

$$S(f) = \frac{1}{\varepsilon}\int_{\mathbf{R}^3} [-(1-\varepsilon f)\log(1-\varepsilon f) - \varepsilon f \log(\varepsilon f)]\,dv \tag{1.3}$$

which is always finite for solutions of Eq. (BFD). Since $0 \leqslant f \leqslant 1/\varepsilon$, the entropy (1.3) has the advantage that the integrands $-(1-\varepsilon f)\log(1-\varepsilon f)$ and $-\varepsilon f\log(\varepsilon f)$ are both nonnegative. The corresponding entropy identity is given by

$$S(f(t)) = S(f_0) + \int_0^t e(f(\tau))\,d\tau, \qquad t \geqslant 0 \tag{1.4}$$

where

$$e(f) = \frac{1}{4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)$$

$$\times\,\Gamma(f'f'_*(1-\varepsilon f)(1-\varepsilon f_*),\, ff_*(1-\varepsilon f')(1-\varepsilon f'_*))\,d\omega\,dv_*\,dv,$$

$$\Gamma(a,b) = \begin{cases} (a-b)\log(a/b), & a>0, b>0; \\ +\infty, & a>0, b=0 \quad \text{or} \quad a=0, b>0; \\ 0, & a=b=0. \end{cases} \tag{1.5}$$

Here and below we denote $f(t) = f(\cdot, t)$.

An equilibrium of Eq. (BFD) is defined to be a time-independent solution of the equation. By entropy identity (1.4) (for $B(\cdot,\cdot) > 0$ a.e.), this is equivalent to say that an equilibrium of Eq. (BFD) is defined to be a solution of the following equation

$$f'f'_*(1-\varepsilon f)(1-\varepsilon f_*)=f\,f_*(1-\varepsilon f')(1-\varepsilon f'_*) \qquad \text{a.e. on } \mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2 \tag{1.6}$$

together with the physical conditions

$$f\in L^1(\mathbf{R}^3),\quad \|f\|_{L^1_0}\neq 0 \qquad \text{and} \qquad 0\leqslant f\leqslant 1/\varepsilon \qquad \text{on } \mathbf{R}^3. \tag{1.7}$$

In our derivation, we often assume that $\varepsilon=1$ in order to simplify notations. In fact, by multiplying ε to both sides of Eq. (BFD) one sees that in Eq. (BFD) the triple (f, B, ε) is equivalent to the triple $(\tilde{f}, \tilde{B}, 1)$ with $\tilde{f}=\varepsilon f$, $\tilde{B}=(1/\varepsilon)\,B$.

The paper is organized as follows. In Section 2, we give some properties of collision integrals. In Section 3 we prove conservation of energy, entropy identity, and give moment production estimates. For spatially inhomogeneous solutions of BFD, the conservation of energy and entropy identity were proven in ref. 9 under the cut-off condition: $B\in L^1(\mathbf{R}^3\times\mathbf{S}^2)$. Uniqueness of conservative solutions of Eq. (BFD) remains unknown for hard potentials. Section 4 gives the classification of equilibria for the BFD model. According to $S(f)>0$ (or $T>\frac{2}{5}T_F$) and $S(f)=0$ (or $T=\frac{2}{5}T_F$), equilibria of Eq. (BFD) are classified to Fermi–Dirac distributions (see (4.5)) and characteristic functions of Euclidean balls respectively. In Section 5 we show that it is the L^∞-bound, $0\leqslant f\leqslant 1/\varepsilon$, that makes the temperatures of the gases can not be very low in comparison with the relevant Fermi temperatures T_F: the inequality $T\geqslant\frac{2}{5}T_F$ holds for all conservative solutions of Eq. (BFD). And we prove that a conservative solution of Eq. (BFD) can only trend towards a Fermi–Dirac distribution unless $T=\frac{2}{5}T_F$ which determines that the solution is a second equilibrium.

2. SOME PROPERTIES OF COLLISION INTEGRALS

Lemma 1. Let $w(t)$ and $\Psi(r)$ be nonnegative Borel functions on $[0,1]$ and $[0,\infty)$ respectively. Let $W(z,\omega)=w(|z|^{-1}\,|\langle z,\omega\rangle|)$. Then for any nonnegative measurable function f on $\mathbf{R}^3$ and for all $v\in\mathbf{R}^3$

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} W(v-v_*,\omega)\,\Psi(|v-v_*|)\,f(v')\,dv_*\,d\omega$$

$$=4\pi\int_0^{\pi/2}\frac{\sin(\theta)\,w(\cos\theta)}{\cos^3\theta}\left\{\int_{\mathbf{R}^3}\Psi\left(\frac{|v-v_*|}{\cos\theta}\right)f(v_*)\,dv_*\right\}d\theta, \tag{2.1}$$

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} W(v-v_*,\omega)\,\Psi(|v-v_*|)\,f(v'_*)\,dv_*\,d\omega$$

$$= 4\pi\int_0^{\pi/2}\frac{\sin(\theta)\,w(\cos\theta)}{\sin^3\theta}\left\{\int_{\mathbf{R}^3}\Psi\left(\frac{|v-v_*|}{\sin\theta}\right)f(v_*)\,dv_*\right\}d\theta. \tag{2.2}$$

Proof. We prove the second equality. The first one is relatively easy. To prove (2.2), we need the following equality which can be easily proven using a spherical coordinate transformation:

$$\int_{\mathbf{S}^2} w(|\langle\sigma,\omega\rangle|)\,\varphi\left(\frac{\sigma-\langle\sigma,\omega\rangle\,\omega}{\sqrt{1-\langle\sigma,\omega\rangle^2}}\right)d\omega$$

$$= 2\int_{\mathbf{S}^2}\frac{\langle\sigma,\omega\rangle}{\sqrt{1-\langle\sigma,\omega\rangle^2}}\,w(\sqrt{1-\langle\sigma,\omega\rangle^2})\,\varphi(\omega)\,1_{\{\langle\sigma,\omega\rangle>0\}}\,d\omega,\quad \forall\sigma\in\mathbf{S}^2 \tag{2.3}$$

where $\varphi(\omega)$ is a nonnegative measurable function on $\mathbf{S}^2$ with respect to the Lebesgue spherical measure $d\omega$.

Making changes of variable $v_* = v + r\sigma$, $r = \rho/\sqrt{1-\langle\sigma,\omega\rangle^2}$ (ω being fixed), and applying (2.3) (with different $w(\cdot)$) deduce that the left-hand side of (2.2) is equal to

$$\int_0^\infty \rho^2\left\{\iint_{\mathbf{S}^2\times\mathbf{S}^2}\frac{w(|\langle\sigma,\omega\rangle|)}{(\sqrt{1-\langle\sigma,\omega\rangle^2}\,)^3}\,\Psi\left(\frac{\rho}{\sqrt{1-\langle\sigma,\omega\rangle^2}}\right)\right.$$

$$\left.\times f\left(v+\rho\left(\frac{\sigma-\langle\sigma,\omega\rangle\,\omega}{\sqrt{1-\langle\sigma,\omega\rangle^2}}\right)\right)d\omega\,d\sigma\right\}d\rho$$

$$= 2\int_0^\infty \rho^2\left\{\iint_{\mathbf{S}^2\times\mathbf{S}^2}\frac{\langle\sigma,\omega\rangle\,w(\sqrt{1-\langle\sigma,\omega\rangle^2}\,)}{\sqrt{1-\langle\sigma,\omega\rangle^2}\,\langle\sigma,\omega\rangle^3}\right.$$

$$\left.\times\Psi\left(\frac{\rho}{\langle\sigma,\omega\rangle}\right)f(v+\rho\omega)\,1_{\langle\sigma,\omega\rangle>0}\,d\omega\,d\sigma\right\}d\rho$$

$$= 4\pi\int_0^{\pi/2}\frac{\cos(\theta)\,w(\sin\theta)}{\cos^3\theta}\left\{\int_0^\infty\int_{\mathbf{S}^2}\rho^2\,\Psi\left(\frac{\rho}{\cos\theta}\right)f(v+\rho\omega)\,d\omega\,d\rho\right\}d\theta$$

$=$ the right-hand side of (2.2). ∎

Lemma 2. Let B be given (or bounded from above) by (1.1) with (1.2). Let $k \geqslant 0$ and $f \in L^1_{k+\beta}(\mathbf{R}^3)$ satisfy $0 \leqslant f \leqslant 1/\varepsilon$.

(a) If $0 \leqslant \beta \leqslant 1$, then for all $\theta_1 \in (0, \pi/4]$ and all $v \in \mathbf{R}^3$

$$\varepsilon \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f' f'_* (1 + |v_*|^2)^{k/2}\, d\omega\, dv_*$$
$$\leqslant 2^{3k+4} A_0 \left(\frac{1}{\sin \theta_1}\right)^{3+\beta} \|f\|_{L^1_{k+\beta}} (1 + |v|^2)^{\beta/2}$$
$$+ 2^{3k+4} A(\theta_1)\, \|f\|_{L^1_0} (1 + |v|^2)^{(k+\beta)/2} \tag{2.4}$$

where

$$A(\theta_1) = \max\left\{4\pi \int_0^{\theta_1} \sin(\theta)\, b(\theta)\, d\theta,\ 4\pi \int_{\pi/2-\theta_1}^{\pi/2} \sin(\theta)\, b(\theta)\, d\theta\right\}. \tag{2.5}$$

(b) If $-3 < \beta \leqslant 0$, then for all $v \in \mathbf{R}^3$

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f' f'_*\, d\omega\, dv_* \leqslant C_1(A_0, \beta, \varepsilon)(\|f\|_{L^1_0})^{(3+\beta)/3}, \tag{2.6}$$

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, f' f'_*\, |v - v_*|^2\, d\omega\, dv_* \leqslant C_2(A_0, \beta, \varepsilon)(1 + \|f\|_{L^1_2})(1 + |v|^2) \tag{2.7}$$

where the constants $C_i(A_0, \beta, \varepsilon)$ depend only on A_0, β and ε.

Proof. We can assume that B is given by (1.1) with (1.2).

(a) Denote $m_s(v) = (1 + |v|^2)^{s/2}$. By $|v_*|^2 \leqslant |v'|^2 + |v'_*|^2$ we have $(m_k)_* \leqslant 2^{k/2}[(m_k)' + (m_k)'_*]$. Then the left-hand side of (2.4) is less than or equal to

$$2^{k/2}\varepsilon \iint_{\mathbf{R}^3 \times \mathbf{S}^2} f' f'_* (m_k)'\, B\, d\omega\, dv_* + 2^{k/2}\varepsilon \iint_{\mathbf{R}^3 \times \mathbf{S}^2} f' f'_* (m_k)'_*\, B\, d\omega\, dv_*. \tag{2.8}$$

Next, by $|v'| \leqslant |v'_*| + |v - v_*|$ and $|v'_*| \leqslant |v'| + |v - v_*|$ we have $(m_k)' \leqslant 2^k[(m_k)'_* + |v - v_*|^k]$ and $(m_k)'_* \leqslant 2^k[(m_k)' + |v - v_*|^k]$ which imply

$$(m_k)'\, B \leqslant (m_k)'\, B_1 + 2^k[(m_k)'_* + |v - v_*|^k]\, B_2, \tag{2.9}$$
$$(m_k)'_*\, B \leqslant (m_k)'_*\, B_3 + 2^k[(m_k)' + |v - v_*|^k]\, B_4, \tag{2.10}$$

where

$$B_1 = B \cdot 1_{\{0 \leqslant \theta < \pi/2 - \theta_1\}}, \qquad B_2 = B \cdot 1_{\{\pi/2 - \theta_1 \leqslant \theta \leqslant \pi/2\}},$$
$$B_3 = B \cdot 1_{\{\theta_1 < \theta \leqslant \pi/2\}}, \qquad B_4 = B \cdot 1_{\{0 \leqslant \theta \leqslant \theta_1\}}$$

and $\theta = \arccos(|\langle v - v_*, \omega\rangle|/|v - v_*|)$. Applying Lemma 1, (2.9) and inequalities

$$|v - v_*|^\beta \leqslant m_\beta \cdot (m_\beta)_*, \qquad |v - v_*|^{k+\beta} \leqslant 2^{k+\beta}[m_{k+\beta} + (m_{k+\beta})_*],$$

we have

$$\begin{aligned}
\varepsilon & \iint_{\mathbf{R}^3 \times \mathbf{S}^2} f' f'_* (m_k)' \, B \, d\omega \, dv_* \\
& \leqslant \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f m_k)' \, B_1 \, d\omega \, dv_* \\
& \quad + 2^k \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f m_k)'_* \, B_2 \, d\omega \, dv_* + 2^k \iint_{\mathbf{R}^3 \times \mathbf{S}^2} f'_* \, |v - v_*|^k \, B_2 \, d\omega \, dv_* \\
& = 4\pi \int_0^{\pi/2 - \theta_1} \frac{\sin(\theta) \, b(\theta)}{(\cos\theta)^{3+\beta}} \, d\theta \int_{\mathbf{R}^3} f(v_*) \, m_k(v_*) \, |v - v_*|^\beta \, dv_* \\
& \quad + 2^k 4\pi \int_{\pi/2 - \theta_1}^{\pi/2} \frac{\sin(\theta) \, b(\theta)}{(\sin\theta)^{3+\beta}} \, d\theta \int_{\mathbf{R}^3} f(v_*) \, m_k(v_*) \, |v - v_*|^\beta \, dv_* \\
& \quad + 2^k 4\pi \int_{\pi/2 - \theta_1}^{\pi/2} \frac{\sin(\theta) \, b(\theta)}{(\sin\theta)^{3+k+\beta}} \, d\theta \int_{\mathbf{R}^3} f(v_*) \, |v - v_*|^{k+\beta} \, dv_* \\
& \leqslant A_0 \left(\frac{1}{\sin\theta_1} \right)^{3+\beta} 2^{(5/2)k+3} \, \|f\|_{L^1_{k+\beta}} \, m_\beta(v) + 2^{(5/2)k+3} A(\theta_1) \, \|f\|_{L^1_0} \, m_{k+\beta}(v).
\end{aligned}$$

Similarly, using (2.10) we have

$$\begin{aligned}
\varepsilon & \iint_{\mathbf{R}^3 \times \mathbf{S}^2} f' f'_* (m_k)'_* \, B \, d\omega \, dv_* \\
& \leqslant A_0 \left(\frac{1}{\sin\theta_1} \right)^{3+\beta} 2^{(5/2)k+3} \, \|f\|_{L^1_{k+\beta}} \, m_\beta(v) + 2^{(5/2)k+3} A(\theta_1) \, \|f\|_{L^1_0} \, m_{k+\beta}(v).
\end{aligned}$$

Combining these with (2.8) give (2.4).

(b) Since $-3 < \beta \leqslant 0$ and $0 \leqslant f \leqslant 1/\varepsilon$, (2.6) and (2.7) are easily derived by splitting $B = B_1 + B_2$ with $\theta_1 = \pi/4$ and using Lemma 1 together with the following estimates (write $\alpha = -\beta$)

$$\int_{\mathbf{R}^3} f_* \, |v - v_*|^{-\alpha} \, dv_* \leqslant C_1(\alpha, \varepsilon)(\|f\|_{L^1_0})^{(3-\alpha)/3},$$

$$\int_{\mathbf{R}^3} f_* \, |v - v_*|^{2-\alpha} \, dv_* \leqslant C_2(\alpha, \varepsilon)(1 + \|f\|_{L^1_2})(1 + |v|^2). \quad \blacksquare$$

Lemma 3. Let B_n, B be collision kernels satisfying for all $(z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2$,

$$0 \leqslant B_n(z, \omega) \leqslant B(z, \omega), \qquad \lim_{n \to \infty} B_n(z, \omega) = B(z, \omega) \tag{2.11}$$

where B is given by (1.1)–(1.2). Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $L_2^1(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$ i.e., $\sup_{n \geqslant 1} \{\|f_n\|_{L_2^1} + \|f_n\|_{L^\infty}\} < \infty$. Suppose that $f_n \rightharpoonup f$ weakly in $L^1(\mathbf{R}^3)$. Then

$$\lim_{n \to \infty} Q_n(f_n)^\wedge(\xi) = Q(f)^\wedge(\xi) \qquad \forall \xi \in \mathbf{R}^3. \tag{2.12}$$

Here $Q_n(f_n)$ and $Q(f)$ are collision integrals corresponding to kernels B_n and B respectively ; $g^\wedge(\xi) = \int_{\mathbf{R}^3} g(v)\, e^{-i\langle \xi, v\rangle}\, dv$ is the Fourier transform.

Proof. Denote $\chi_\xi(v) = e^{-i\langle \xi, v\rangle}$. Observe that the four-product term $f f_* f' f'_*$ can be canceled from the collision integral $Q(f)$. We have (after suitable changes of integral variables)

$$Q_n(f_n)^\wedge(\xi) = \sum_{j=1}^{6} \mathcal{Q}_j^{B_n}(f_n)(\xi), \qquad \xi \in \mathbf{R}^3 \tag{2.13}$$

where $\mathcal{Q}_j^{\{\cdot\}}(\cdot)$ are defined by

$$\mathcal{Q}_1^B(f)(\xi) = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} f(v)\, f(v_*) \left(\int_{\mathbf{S}^2} B(v - v_*, \omega)\, \chi_\xi(v')\, d\omega \right) dv_*\, dv,$$

$$\mathcal{Q}_2^B(f)(\xi) = -\iint_{\mathbf{R}^3 \times \mathbf{R}^3} (f\chi_\xi)(v)\, f(v_*) \left(\int_{\mathbf{S}^2} B(v - v_*, \omega)\, d\omega \right) dv_*\, dv,$$

$$\mathcal{Q}_3^B(f)(\xi) = -\varepsilon \int_{\mathbf{R}^3} (f\chi_\xi)(v)\, Q^+(f, f)(v)\, dv,$$

$$\mathcal{Q}_4^B(f)(\xi) = -\varepsilon \int_{\mathbf{R}^3} f(v)\, \chi_\xi(-v)\, Q^+(f\chi_\xi, f\chi_\xi)(v)\, dv,$$

$$\mathcal{Q}_5^B(f)(\xi) = \varepsilon \int_{\mathbf{R}^3} f(v)\, Q^+(f\chi_\xi, f)(v)\, dv,$$

$$\mathcal{Q}_6^B(f)(\xi) = \varepsilon \int_{\mathbf{R}^3} f(v)\, Q^+(f, f\chi_\xi)(v)\, dv,$$

and $Q^+(\cdot,\cdot)$ is the usual "gain" term of the Boltzmann's collision operator:

$$Q^+(f,g)(v)=\iint_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\, f(v')\, g(v'_*)\, d\omega\, dv_*.$$

It should be noted that for $\mathcal{Q}_4^B(f)(\xi)$ we have used the following decomposition:

$$\chi_\xi(v_*)=\chi_\xi(-v)\,\chi_\xi(v')\,\chi_\xi(v'_*).$$

From the structures of $\mathcal{Q}_j^B(f)$ we obtain the following convergence:

$$\lim_{n\to\infty}\mathcal{Q}_j^{B_n}(f_n)(\xi)=\mathcal{Q}_j^B(f)(\xi),\qquad \xi\in\mathbf{R}^3,\quad j=1,2,\ldots,6. \tag{2.14}$$

In fact, (2.14) is obvious for $j=1,2$; for $j=3,4,5,6$, (2.14) is a consequence of a well-known result of P. L. Lions about the compactness of the gain term $Q^+(f,g)$.[11, 12] Therefore (2.12) follows from (2.13) and (2.14). ∎

3. CONSERVATION OF ENERGY, ENTROPY IDENTITY, AND MOMENT PRODUCTION ESTIMATES

For completeness, we first give here a short proof for the existence and uniqueness of conservative solutions of Eq. (BFD) in the case of non-hard potentials: $B(z,\omega)\leqslant b(\theta)\,|z|^\beta$, $-3<\beta\leqslant 0$ where $b(\theta)$ satisfies (1.2). Suppose $\varepsilon=1$. Given $f_0\in L_2^1(\mathbf{R}^3)$ with $0\leqslant f_0\leqslant 1$. For any $\delta>0$, let $\mathcal{B}_\delta$ be the collection of measurable functions $f\in L^\infty([0,\delta];L_2^1(\mathbf{R}^3))$ satisfying $\|f\|_\delta := \sup_{t\in[0,\delta]}\|f(t)\|_{L_2^1}\leqslant 2\,\|f_0\|_{L_2^1}$. Denote $a\wedge b=\min\{a,b\}$. Let $J(f)(v,t)=f_0(v)+\int_0^t Q(|f|\wedge 1)(v,\tau)\,d\tau$. By Lemma 2 Part(b), there is a small $\delta>0$ which depends only on A_0, β and $\|f_0\|_{L_2^1}$, such that J is a contraction mapping from the complete metric space $(\mathcal{B}_\delta, \|\cdot-\cdot\|_\delta)$ into itself. Thus there exists a unique $f\in\mathcal{B}_\delta$ such that $\|f-J(f)\|_\delta=0$. After a modification on v-null sets, there is a null set $Z_\delta\subset\mathbf{R}^3$ such that $f(v,t)=J(f)(v,t)$ for all $t\in[0,\delta]$ and all $v\in\mathbf{R}^3\setminus Z_\delta$. Next, we have (denote $(y)^+=\max\{y,0\}$)

$$(-f(v,t))^+\leqslant\int_0^t Q^-(|f|\wedge 1)(v,\tau)\,1_{\{f(v,\tau)<0\}}\,d\tau,\qquad t\in[0,\delta],\quad v\in\mathbf{R}^3\setminus Z_\delta$$

and so by Gronwall lemma we obtain $(-f(v,t))^+=0$. Also, we have

$$(f(v,t)-1)^+\leqslant\int_0^t Q^+(|f|\wedge 1)(v,\tau)\,1_{\{f(v,\tau)>1\}}\,d\tau=0.$$

Therefore $0 \leqslant f \leqslant 1$ on $(\mathbf{R}^3 \backslash Z_\delta) \times [0, \delta]$. After modifications on v-null sets, f is a unique conservative solution of Eq. (BED) on $\mathbf{R}^3 \times [0, \delta]$. By conservation of mass and energy, we have $\|f(\delta)\|_{L^1_2} = \|f_0\|_{L^1_2}$. Thus with the same $\delta > 0$ and replacing the initial f_0 by $f(\cdot, \delta)$, $f(\cdot, 2\delta)$,..., respectively, the solution f can be inductively extended to all intervals $[\delta, 2\delta]$, $[2\delta, 3\delta]$,..., and the extended function f is a unique conservative solution of Eq. (BFD) on $\mathbf{R}^3 \times [0, \infty)$. Existence for hard potentials follows from this result (with $\beta = 0$) and a weak stability property (see Proposition 1 and Theorem 2 below).

Theorem 1. Suppose the kernel B is given (or bounded from above) by (1.1) with (1.2). Let $f_0 \in L^1_2(\mathbf{R}^3)$ satisfy $0 \leqslant f_0 \leqslant 1/\varepsilon$, and let f be any solution of Eq. (BFD) with $f|_{t=0} = f_0$. Then

(1) If $-3 < \beta \leqslant 0$, or, if $0 < \beta \leqslant 1$ and $\int_{\mathbf{R}^3} f(v, t)\, |v|^2\, dv \leqslant \int_{\mathbf{R}^3} f_0(v)\, |v|^2\, dv$ for all $t \geqslant 0$, then f conserves the energy and therefore f is a conservative solution.

(2) The entropy identity (1.4) does actually hold. Moreover if $f \in L^\infty([0, \infty); L^1_2(\mathbf{R}^3))$, then $\sup_{t \geqslant 0} S(f(t)) < \infty$.

Proof. Suppose $\varepsilon = 1$. For $-3 < \beta \leqslant 0$, we have proved in above that the solution is unique and conserves the energy. For $0 < \beta \leqslant 1$, our proof for conservation of energy is completely the same to that for the original Boltzmann equation,[13] so we omit it here. Now we prove the entropy identity (1.4). First of all, the entropy $S(f(t))$ is finite for all $t \geqslant 0$. In fact for any $g \in L^1_2(\mathbf{R}^3)$ with $0 \leqslant g \leqslant 1$ we have

$$(1-g)\, |\log(1-g)| + g\, |\log g| \leqslant g(1+|v|^2) + e^{-(1/2)\,|v|^2}, \qquad v \in \mathbf{R}^3. \tag{3.1}$$

This also implies that if $f \in L^\infty([0, \infty); L^1_2(\mathbf{R}^3))$, then $\sup_{t \in [0, \infty)} S(f(t)) < \infty$. Next, let $\phi(v) = e^{-|v|}$, $\phi_n(v) = (1/n)\, \phi(v)$ ($n \in \mathbf{N}$, the set of positive integers), and let

$$\Psi_n(f) = -(1 - f + \phi_n) \log(1 - f + \phi_n) - (f + \phi_n) \log(f + \phi_n),$$

$$S_n(f(t)) = \int_{\mathbf{R}^3} \Psi_n(f)(v, t)\, dv.$$

It is easily shown that for all $n \in \mathbf{N}$,

$$|\Psi_n(f)(v, t)| \leqslant 3[f(v, t) + \phi(v)](1 + |v|^2) + e^{-(1/2)\,|v|^2}.$$

This gives $\lim_{n\to\infty} S_n(f(t)) = S(f(t))$ by dominated convergence theorem. Since $\phi_n(v) > 0$ and $t \mapsto f(v, t)$ is absolutely continuous, we have for all $v \in \mathbf{R}^3 \setminus Z$ (mes$(Z) = 0$)

$$\Psi_n(f)(v, t) = \Psi_n(f_0)(v) - \int_0^t Q(f)(v, \tau) \times \log\left(\frac{f(v, \tau) + \phi_n(v)}{1 - f(v, \tau) + \phi_n(v)}\right) d\tau, \qquad t \geqslant 0.$$

Next, we have, for some constants $C_n > 0$, $|\log[(f + \phi_n)/(1 - f + \phi_n)]| \leqslant C_n(1 + |v|)$. This implies that $Q^{\pm}(f) \log[(f + \phi_n)/(1 - f + \phi_n)] \in L^1(\mathbf{R}^3 \times [0, t_1])$ for all $t_1 > 0$. Thus by classical derivation[5, 20] we obtain

$$S_n(f(t)) = S_n(f_0) + \int_0^t e_n(f(\tau))\, d\tau \tag{3.2}$$

where

$$e_n(f(\tau)) = \frac{1}{4} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\, \Gamma_n(f)(v, v_*, \omega, \tau)\, d\omega\, dv_*\, dv,$$

$$\Gamma_n(f)(v, v_*, \omega, \tau) = [f' f'_* (1 - f)(1 - f_*) - f f_* (1 - f')(1 - f'_*)] \times \log\left(\frac{(f + \phi_n)'(f + \phi_n)'_*\, (1 - f + \phi_n)(1 - f + \phi_n)_*}{(f + \phi_n)(f + \phi_n)_*\, (1 - f + \phi_n)'\, (1 - f + \phi_n)'_*}\right).$$

Let

$$e_n^+(f(\tau)) = \tfrac{1}{4} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)[\Gamma_n(f)(v, v_*, \omega, \tau)]^+\, d\omega\, dv_*\, dv,$$

$$e_n^-(f(\tau)) = \tfrac{1}{4} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)[-\Gamma_n(f)(v, v_*, \omega, \tau)]^+\, d\omega\, dv_*\, dv.$$

Then (3.2) is written

$$\int_0^t e_n^+(f(\tau))\, d\tau = S_n(f(t)) - S_n(f_0) + \int_0^t e_n^-(f(\tau))\, d\tau. \tag{3.3}$$

It is easily seen that for all $(v, v_*, \omega, \tau) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2 \times [0, \infty)$

$$\lim_{n\to\infty} [\Gamma_n(f)(v, v_*, \omega, \tau)]^+ = \Gamma(f' f'_* (1 - f)(1 - f_*), f f_* (1 - f')(1 - f'_*)),$$

$$\lim_{n\to\infty} [-\Gamma_n(f)(v, v_*, \omega, \tau)]^+ = 0$$

where $\Gamma(\cdot,\cdot)$ is the function (1.5). Moreover applying the elementary inequalities

$$[(a-b)\log(a_1/b_1)]^+ \leqslant \Gamma(a,b)+a_1-a+b_1-b,$$
$$[-(a-b)\log(a_1/b_1)]^+ \leqslant a_1-a+b_1-b$$

for $0\leqslant a<a_1$, $0\leqslant b<b_1$, we obtain the following controls:

$$[\Gamma_n(f)(v,v_*,\omega,\tau)]^+ \leqslant \Gamma(f'f'_*(1-f)(1-f_*)\,,\, ff_*(1-f')(1-f'_*)) \\ +4(f+\phi)(f+\phi)_*+4(f+\phi)'\,(f+\phi)'_*,$$
$$[-\Gamma_n(f)(v,v_*,\omega,\tau)]^+ \leqslant 4(f+\phi)(f+\phi)_*+4(f+\phi)'\,(f+\phi)'_*.$$

Thus by dominated convergence theorem we obtain for all $t\geqslant 0$

$$\lim_{n\to\infty}\int_0^t e_n^-(f(\tau))\,d\tau=0,$$

and (by (3.3))

$$\lim_{n\to\infty}\int_0^t e_n^+(f(\tau))\,d\tau=S(f(t))-S(f_0). \tag{3.4}$$

By Fatou's Lemma, (3.4) gives

$$\int_0^t e(f(\tau))\,d\tau \leqslant S(f(t))-S(f_0)<\infty \qquad \forall t\in[0,\infty). \tag{3.5}$$

This integrability together with (3.4) and dominated convergence imply that the equality sign in (3.5) holds for all $t\geqslant 0$, i.e., f satisfies the entropy identity (1.4). ∎

To obtain moment production estimates we need a weak stability of the BFD model.

Proposition 1. Let B_n, B be collision kernels satisfying (2.11) and B is given by (1.1)–(1.2). Given initial data f_0^n, f_0 satisfying $0\leqslant f_0^n, f_0\leqslant 1/\varepsilon$, $f_0^n, f_0\in L_2^1(\mathbf{R}^3)$ and $\lim_{n\to\infty}\|f_0^n-f_0\|_{L_2^1}=0$. Let f^n be conservative solutions of Eq. (BFD) corresponding to kernels B_n and $f^n|_{t=0}=f_0^n$. Then there exist a subsequence $\{f^{n_k}\}_{k=1}^{\infty}$ and a conservative solution f of Eq. (BFD) corresponding to the kernel B and $f|_{t=0}=f_0$, such that

$$f^{n_k}(\cdot,t)\rightharpoonup f(\cdot,t) \qquad \text{weakly in } L^1(\mathbf{R}^3) \qquad (k\to\infty) \quad \forall t\in[0,\infty).$$

Proof. We have, for some constant C depending only on A_0, β, ε and $\sup_{n\geqslant 1}\|f_0^n\|_{L_2^1}$,

$$\sup_{n\geqslant 1}\|f^n(t_1)-f^n(t_2)\|_{L_0^1}\leqslant C\,|t_1-t_2|,\qquad t_1,t_2\in[0,\infty).$$

Since $\{f^n(\cdot,t)\}_{n=1}^{\infty}$ is weakly compact in $L^1(\mathbf{R}^3)$ for all $t\geqslant 0$, the standard diagonal process and the condition $\lim_{n\to\infty}\|f_0^n-f_0\|_{L_2^1}=0$ deduce that there exists a common subsequence, still denote it by $\{f^n(\cdot,t)\}$, such that for every $t\in[0,\infty)$, $f^n(\cdot,t)$ converges weakly in $L^1(\mathbf{R}^3)$ to some $f(\cdot,t)\in L^1(\mathbf{R}^3)$ $(n\to\infty)$ and f is measurable on $\mathbf{R}^3\times[0,\infty)$ satisfying $0\leqslant f\leqslant 1$, $f|_{t=0}=f_0$, $\|f(t)\|_{L_0^1}=\|f_0\|_{L_0^1}$ and $\int_{\mathbf{R}^3}f(v,t)\,|v|^2\,dv\leqslant\int_{\mathbf{R}^3}f_0(v)\,|v|^2\,dv$ for all $t\geqslant 0$. To prove that f is a solution of Eq. (BFD), we consider the Fourier transform: Let $J(f)(v,t)=f_0(v)+\int_0^t Q(f)(v,\tau)\,d\tau$. We have for all $\xi\in\mathbf{R}^3$

$$J(f)(\cdot,t)^\wedge(\xi)=f_0{}^\wedge(\xi)+\int_0^t Q(f)(\cdot,\tau)^\wedge(\xi)\,d\tau,$$

$$f^n(\cdot,t)^\wedge(\xi)=f_0^n{}^\wedge(\xi)+\int_0^t Q_n(f^n)(\cdot,\tau)^\wedge(\xi)\,d\tau.$$

Since $\sup_{n\geqslant 1,t\geqslant 0}\|f^n(t)\|_{L_2^1}=\sup_{n\geqslant 1}\|f_0^n\|_{L_2^1}<\infty$, it is easily seen from the representation (2.13) and from Lemma 2 (with $k=0$, $\theta_1=\pi/4$ in case $0<\beta\leqslant 1$) that $\sup_{n\geqslant 1,\tau\geqslant 0}|Q_n(f^n)(\cdot,\tau)^\wedge(\xi)|<\infty$ for all $\xi\in\mathbf{R}^3$. Thus by Lemma 3 we have

$$f(\cdot,t)^\wedge(\xi)=J(f)(\cdot,t)^\wedge(\xi)\qquad\forall t\geqslant 0,\quad\forall\xi\in\mathbf{R}^3.$$

Therefore for all $t\geqslant 0$, $f(v,t)=J(f)(v,t)$ a.e. $v\in\mathbf{R}^3$. After modifications on v-null sets, f is a solution of Eq. (BFD) and conserves the mass and momentum. The conservation of energy follows from Theorem 1. ∎

Now we give the moment production estimates of Wennberg's type.[22, 13, 14]

Theorem 2. Suppose the kernel B satisfies (1.1)–(1.2) with $0\leqslant\beta\leqslant 1$. Let $f_0\in L_2^1(\mathbf{R}^3)$ satisfy $0\leqslant f_0\leqslant 1/\varepsilon$ and $\|f_0\|_{L_2^1}>0$. Then there exists a conservative solution f of Eq. (BFD) with $f|_{t=0}=f_0$ such that

(I) If $\beta>0$, then for any $s>2$

$$\|f(t)\|_{L_s^1}\leqslant\left[\frac{b}{1-\exp(-at)}\right]^{(s-2)/\beta}\qquad\forall t>0$$

where $a>0$, $b>0$ are constants depending only on β, s, $\|f_0\|_{L^1_0}$, $\|f_0\|_{L^1_2}$, and on some integration of $b(\theta)$. In particular, a, b do not depend on the parameter ε.

(II) If $\beta=0$ and $f_0\in L^1_s(\mathbf{R}^3)$ for some $2<s\leqslant 4$, then $f\in L^\infty([0,\infty); L^1_s(\mathbf{R}^3))$.

Proof. We first assume that $f_0\in L^1_s(\mathbf{R}^3)$ for all $s\geqslant 2$. For any $k\in\mathbf{N}$, let $B_k(z,\omega)\equiv b(\theta)(|z|\wedge k)^\beta$ and let f_k be conservative solutions of Eq. (BFD) corresponding to $B_k(z,\omega)$ with $f_k|_{t=0}=f_0$. Existence of the solutions f_k has been shown above since $B_k(z,\omega)\leqslant k^\beta b(\theta)$. In the following, we will use the function $m_s(v)=(1+|v|^2)^{s/2}$. Consider $\phi_n(v)=m_s(v)\wedge n$, $n\in\mathbf{N}$. By the inequality $|v'|^2\leqslant |v|^2+|v_*|^2$ we have $\phi_n'\leqslant 2^{s/2-1}[\phi_n+\phi_{n*}]$. This gives

$$\|f_k(t)\,\phi_n\|_{L^1_0}\leqslant \|f_0\|_{L^1_s}+2^{s/2}A_0\,k^\beta\,\|f_0\|_{L^1_0}\int_0^t \|f_k(\tau)\,\phi_n\|_{L^1_0}\,d\tau,\qquad t\geqslant 0.$$

Thus using Gronwall lemma and then letting $n\to\infty$ leads to $f_k\in L^\infty_{\rm loc}([0,\infty); L^1_s(\mathbf{R}^3))$ for all $s>2$. Therefore using the Povzner's inequality (see, e.g., ref. 5)

$$(m_s)'+(m_s)'_*-m_s-(m_s)_*\leqslant 2^s[m_{s-1}(m_1)_*+m_1(m_{s-1})_*]$$

we obtain

$$\|f_k(t)\|_{L^1_s}\leqslant \|f_0\|_{L^1_s}+2^sA_0\|f_0\|_{L^1_2}\int_0^t \|f_k(\tau)\|_{L^1_s}\,d\tau,\qquad t\geqslant 0$$

and so

$$\|f_k(t)\|_{L^1_s}\leqslant \|f_0\|_{L^1_s}\exp\{2^sA_0\|f_0\|_{L^1_2}\,t\},\qquad t\geqslant 0. \tag{3.6}$$

By weak stability (Proposition 1), there exists a conservative solution f of Eq. (BFD) corresponding to B with $f|_{t=0}=f_0$ such that for any $t\geqslant 0$, $f(\cdot,t)$ is an L^1-weak limit of a common subsequence of $\{f_k(\cdot,t)\}_{k=1}^\infty$. Taking the weak limit, (3.6) also holds for f and so $f\in L^\infty_{\rm loc}([0,\infty); L^1_s(\mathbf{R}^3))$ for all $s\geqslant 2$. By calculation using Lemma 2 Part (a) (with $\theta_1=\pi/4$), the high-moment property of f implies that $Q^\pm(f)\in L^\infty_{\rm loc}([0,\infty); L^1_s(\mathbf{R}^3))$ and $Q(f)\in {\rm Lip}([0,t_1]; L^1_s(\mathbf{R}^3))$ for all $s\geqslant 2$ and all $t_1>0$. Thus for all $s>2$, $f\in C^1([0,\infty); L^1_s(\mathbf{R}^3))$. Then, using a sharpened version of the Povzner's inequality (see ref. 13 and the proof therein)

$$\begin{aligned}&(m_s)'+(m_s)'_*-m_s-(m_s)_*\\&\quad\leqslant 2(2^{s/2}-2)[m_{s-\gamma}(m_\gamma)_*+m_\gamma(m_{s-\gamma})_*]-2^{-s-1}(s-2)[\kappa(\theta)]^s\,m_s\end{aligned}$$

where $\kappa(\theta)=\min\{\cos\theta, 1-\cos\theta\}$, $\theta=\arccos(|v-v_*|^{-1}|\langle v-v_*, \omega\rangle|)$, $0\leqslant\gamma\leqslant\min\{s/2, 2\}$, $s>2$, we obtain for all $t\geqslant 0$

$$\begin{aligned}
\frac{d}{dt}\|f(t)\|_{L^1_s} &= \int_{\mathbf{R}^3} Q(f)(v,t)\, m_s(v)\, dv \\
&= \frac{1}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B f f_* [(m_s)' + (m_s)'_* - m_s - (m_s)_*]\, d\omega\, dv_*\, dv \\
&\quad + \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\varepsilon f f' f'_* [(m_s)' + (m_s)'_* - m_s - (m_s)_*]\, d\omega\, dv_*\, dv \\
&\leqslant 2(2^{s/2}-2)\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B f f_*\, m_{s-\gamma}(m_\gamma)_*\, d\omega\, dv_*\, dv \\
&\quad - 2^{-s-2}(s-2)\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\,[\kappa(\theta)]^s f f_*\, m_s\, d\omega\, dv_*\, dv \\
&\quad + 2(2^{s/2}-2)\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\varepsilon f f' f'_* [m_{s-\gamma}(m_\gamma)_* \\
&\quad + m_\gamma(m_{s-\gamma})_*]\, d\omega\, dv_*\, dv \\
&\leqslant (s-2)[2^s I_{s,1}(t) - 2^{-s-2} I_{s,2}(t) + 2^s I_{s,3}(t)]
\end{aligned} \tag{3.7}$$

where $I_{s,j}(t)$ $(j=1,2,3)$ denote the last three integrals.

(I) $\beta>0$. By $\beta\leqslant 1$, we have $|v-v_*|^\beta\geqslant m_\beta(v)-m_\beta(v_*)$. Choose $\gamma=\beta$. Since f conserves the mass and energy, these imply

$$I_{s,1}(t)\leqslant A_0\,\|f_0\|_{L^1_2}\,\|f(t)\|_{L^1_s},$$

$$I_{s,2}(t)\geqslant A_s\,\|f_0\|_{L^1_0}\,\|f(t)\|_{L^1_{s+\beta}} - A_s\,\|f_0\|_{L^1_2}\,\|f(t)\|_{L^1_s},$$

where

$$A_s = 4\pi\int_0^{\pi/2}\sin(\theta)\, b(\theta)[\kappa(\theta)]^s\, d\theta.$$

Also, by Lemma 2 (2.4) (with $k=\beta$ and $k=s-\beta$ respectively), we have for any $\theta_1\in(0,\pi/4]$

$$I_{s,3}(t) = \int_{\mathbf{R}^3} f(1+|v|^2)^{(s-\beta)/2} \left\{ \varepsilon \iint_{\mathbf{R}^3\times\mathbf{S}^2} B f' f'_*(1+|v_*|^2)^{\beta/2}\, d\omega\, dv_* \right\} dv$$

$$+ \int_{\mathbf{R}^3} f(1+|v|^2)^{\beta/2} \left\{ \varepsilon \iint_{\mathbf{R}^3\times\mathbf{S}^2} B f' f'_*(1+|v_*|^2)^{(s-\beta)/2}\, d\omega\, dv_* \right\} dv$$

$$\leqslant 2^{3s+5} A_0 \left(\frac{1}{\sin\theta_1}\right)^{3+\beta} \|f_0\|_{L^1_2}\, \|f(t)\|_{L^1_s} + 2^{3s+5} A(\theta_1)\, \|f_0\|_{L^1_0}\, \|f(t)\|_{L^1_{s+\beta}}.$$

Here $A(\theta)$ is the continuous function (2.5). Thus by (3.7)

$$\frac{d}{dt}\|f(t)\|_{L^1_s} \leqslant (s-2)[2^s A_0 + 2^{-s-2} A_s + 2^{4s+5} A_0 (\sin\theta_1)^{-4}]\, \|f_0\|_{L^1_2}\, \|f(t)\|_{L^1_s}$$

$$-(s-2)[2^{-s-2} A_s - 2^{4s+5} A(\theta_1)]\, \|f_0\|_{L^1_0}\, \|f(t)\|_{L^1_{s+\beta}}.$$

Since $A(\pi/4) \geqslant \frac{1}{2} A_s > 0 = A(0)$, there exists $0 < \theta_1 < \pi/4$ such that $2^{-s-2} A_s - 2^{4s+5} A(\theta_1) = 2^{-s-3} A_s$. Also, we have $\|f(t)\|_{L^1_{s+\beta}} \geqslant [\|f_0\|_{L^1_2}]^{-\beta/(s-2)}$ $[\|f(t)\|_{L^1_s}]^{1+\beta/(s-2)}$ by Hölder inequality. Thus

$$\frac{d}{dt}\|f(t)\|_{L^1_s} \leqslant (s-2)\, C_{s,1}\, \|f(t)\|_{L^1_s} - (s-2)\, C_{s,2} [\|f(t)\|_{L^1_s}]^{1+\beta/(s-2)}$$

which implies

$$\|f(t)\|_{L^1_s} \leqslant \left[\frac{b_s}{1-\exp(-a_s\, t)}\right]^{(s-2)/\beta}, \qquad t>0 \tag{3.8}$$

where $a_s = \beta C_{s,1} > 0$, $b_s = C_{s,1}/C_{s,2} > 0$ depend only on $((\|f_0\|_{L^1_0})^{-1}, \|f_0\|_{L^1_2},$ $A_0, A_s, s, \beta)$.

(II) $\beta = 0$ and $2 < s \leqslant 4$. In this case we can choose $\gamma = s/2$. Then

$$I_{s,1}(t) = A_0(\|f(t)\|_{L^1_{s/2}})^2 \leqslant A_0(\|f_0\|_{L^1_2})^2, \qquad I_{s,2}(t) = A_s\, \|f_0\|_{L^1_0}\, \|f(t)\|_{L^1_s},$$

and by Lemma 2 (with $k = s/2$, $\beta = 0$)

$$I_{s,3}(t) = 2\int_{\mathbf{R}^3} f(1+|v|^2)^{s/4} \left\{ \varepsilon \iint_{\mathbf{R}^3\times\mathbf{S}^2} B f' f'_*(1+|v_*|^2)^{s/4}\, d\omega\, dv_* \right\} dv$$

$$\leqslant 2^{2s+5} A_0 \left(\frac{1}{\sin\theta_1}\right)^3 (\|f_0\|_{L^1_2})^2 + 2^{2s+5} A(\theta_1)\, \|f_0\|_{L^1_0}\, \|f(t)\|_{L^1_s}.$$

Therefore by (3.7)

$$\frac{d}{dt}\|f(t)\|_{L^1_s} \leqslant (s-2)[2^s+2^{3s+5}(\sin\theta_1)^{-3}]\, A_0(\|f_0\|_{L^1_2})^2$$
$$-(s-2)[2^{-s-2}A_s-2^{3s+5}A(\theta_1)]\, \|f_0\|_{L^1_0}\, \|f(t)\|_{L^1_s}, \qquad t\geqslant 0. \tag{3.9}$$

Choose $0<\theta_1<\pi/4$ such that $2^{-s-2}A_s-2^{3s+5}A(\theta_1)=2^{-s-3}A_s$. Then (3.9) implies that with the constant $C_s=[2^s+2^{3s+5}(\sin\theta_1)^{-3}]\, A_0(\|f_0\|_{L^1_2})^2/[2^{-s-3}A_s\, \|f_0\|_{L^1_0}]$,

$$\|f(t)\|_{L^1_s} \leqslant \|f_0\|_{L^1_s}+C_s, \qquad t\geqslant 0. \tag{3.10}$$

Now let f_0 be given in the theorem. Let $f^n_0(v)=f_0(v)\, e^{-(1/n)\,|v|^2}$, and let f^n be conservative solutions of Eq. (BFD) obtained in the above argument with $f^n|_{t=0}=f^n_0$, such that (f^n, f^n_0) satisfy the estimates (3.8) for $\beta>0$ and (3.10) for $\beta=0$ respectively. Since in (3.8) and (3.10) for f^n the coefficients a_s, b_s and C_s depend only on $((\|f^n_0\|_{L^1_0})^{-1}, \|f^n_0\|_{L^1_2}, A_0, A_s, s, \beta)$ and are continuous with respect to $((\|f^n_0\|_{L^1_0})^{-1}, \|f^n_0\|_{L^2_0})$, the conclusion of the theorem follows by taking weak limit and applying Proposition 1. ∎

4. CLASSIFICATION OF EQUILIBRIA

We need the following result which gives a new characterization of the Euclidean n-ball in terms of an equilibrium state of the BFD model.

Proposition 2. Let $n\geqslant 2$, let K be a compact set in $\mathbf{R}^n$ with $\mathrm{mes}(K)>0$ and satisfy

$$1_K(v)\, 1_K(v_*)\left[1-1_K\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\,\omega\right)\right]\left[1-1_K\left(\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\,\omega\right)\right]=0 \tag{4.1}$$

for all $(v, v_*, \omega)\in \mathbf{R}^n\times\mathbf{R}^n\times\mathbf{S}^{n-1}$. Then K is a convex body of constant width. Moreover if $n\geqslant 3$, then K is a Euclidean n-ball.

Our proof of this result is based on the following classical characterization:

Theorem MSW. Let $n\geqslant 3$, let $K\subset\mathbf{R}^n$ be an n-dimensional convex body (i.e., n-dimensional compact convex set) and let p_0 be an interior point of K with the property that for every $n-1$-dimensional plane Π

of $\mathbf{R}^n$ through p_0, the intersection $\Pi \cap K$ is an $n-1$-dimensional convex body of constant width. Then K is a Euclidean n-ball.

Theorem MSW is a special version of a result of Motejano.[15] For $n=3$ see also Süss[18] (under differentiability conditions) and Wegner.[21]

For a set $E \subset \mathbf{R}^n$, let ∂E denote the boundary of E, and let $E^\circ = E \setminus \partial E$.

Proof of Proposition 2. *Step 1.* Let $\operatorname{conv}(K)$ be the convex hull of K. Since $\operatorname{mes}(K) > 0$, $\operatorname{conv}(K)$ is an n-dimensional convex body. In this step we prove that $\partial(\operatorname{conv}(K)) \subset \partial K$. Given any $v_0 \in \partial(\operatorname{conv}(K))$, there is an $\omega \in \mathbf{S}^{n-1}$ and a supporting plane $H^{(-)} = \{v \in \mathbf{R}^n \mid \langle v - v_0, \omega \rangle = 0\}$ of $\operatorname{conv}(K)$ such that

$$\langle v - v_0, \omega \rangle \leqslant 0 \qquad \forall v \in \operatorname{conv}(K). \tag{4.2}$$

For $H^{(-)}$, there is a parallel supporting plane $H^{(+)} = \{v \in \mathbf{R}^n \mid \langle v - u_0, \omega \rangle = 0\}$ of $\operatorname{conv}(K)$ with $u_0 \in \partial(\operatorname{conv}(K))$ and $u_0 \neq v_0$, such that

$$\langle v - u_0, \omega \rangle \geqslant 0 \qquad \forall v \in \operatorname{conv}(K). \tag{4.3}$$

Let Γ be the set of all extrem points of $\operatorname{conv}(K)$. Then $\Gamma \subset \partial K \subset K$. Let

$$d = \max\{|u - v| \mid u \in H^{(-)} \cap \operatorname{conv}(K),\, v \in H^{(+)} \cap \operatorname{conv}(K)\}.$$

For any $u_1 \in H^{(-)} \cap \operatorname{conv}(K)$ and any $v_1 \in H^{(+)} \cap \operatorname{conv}(K)$ satisfying $|u_1 - v_1| = d$, it is easily seen (use (4.2), (4.3)) that $u_1, v_1 \in \Gamma$ and thus $u_1, v_1 \in K$. We assert that $|u_1 - v_1| = \langle u_1 - v_1, \omega \rangle$. This will prove that $v_0 \in \partial K$. In fact, this equality implies that $d = |u_1 - v_1| = \langle u_1 - v_0, \omega \rangle + \langle v_0 - v_1, \omega \rangle = \langle v_0 - v_1, \omega \rangle \leqslant |v_0 - v_1| \leqslant d$ and so $|v_0 - v_1| = d$ which implies that $v_0 \in \Gamma \subset \partial K$. Now suppose, to the contrary, that $|u_1 - v_1| > \langle u_1 - v_1, \omega \rangle$. Then, since $u_1 \in H^{(-)}$ and $v_1 \in H^{(+)}$, we have

$$\left\langle \frac{u_1 + v_1}{2} + \frac{|u_1 - v_1|}{2}\,\omega - v_0\,,\, \omega \right\rangle = \frac{1}{2}\,|u_1 - v_1| - \frac{1}{2}\,\langle u_1 - v_1, \omega \rangle > 0,$$

$$\left\langle \frac{u_1 + v_1}{2} - \frac{|u_1 - v_1|}{2}\,\omega - u_0\,,\, \omega \right\rangle = \frac{1}{2}\,\langle u_1 - v_1, \omega \rangle - \frac{1}{2}\,|u_1 - v_1| < 0.$$

By (4.2) and (4.3) we see that both $\frac{1}{2}(u_1 + v_1) + \frac{1}{2}|u_1 - v_1|\,\omega$ and $\frac{1}{2}(u_1 + v_1) - \frac{1}{2}|u_1 - v_1|\,\omega$ do not belong to K. Since $u_1, v_1 \in K$, this contradicts Eq. (4.1).

Step 2. We prove that $\operatorname{conv}(K) = K$. Since $\operatorname{conv}(K)$ is a convex body, it suffices to show that $(\operatorname{conv}(K))^\circ \subset K$. Given any $x \in (\operatorname{conv}(K))^\circ$. Let $a \in \operatorname{conv}(K)$ satisfy $|a - x| = \max\{|v - x| \mid v \in \operatorname{conv}(K)\}$. Let $H = \{v \in \mathbf{R}^n \mid \langle v - x,$

$a-x\rangle = 0\}$. Choose $b \in H \cap \mathrm{conv}(K)$ such that $|b-x| = \max\{|v-x| \mid v \in H \cap \mathrm{conv}(K)\}$. It is easily verified that $a, b \in \partial(\mathrm{conv}(K))$ and $|a+b-2x|^2 = |a-x|^2 + |b-x|^2 = |a-b|^2$. Take $\omega = \frac{a+b-2x}{|a-b|}$. Then $x = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|\,\omega$. Let $y = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|\,\omega$. Then $|y-x|^2 = |a-b|^2 > |a-x|^2$ and so $y \notin \mathrm{conv}(K)$ therefore $y \notin K$. But the Step 1 shows that $a, b \in \partial K \subset K$, so by Eq. (4.1) we must have $x \in K$. This proves $(\mathrm{conv}(K))^\circ \subset K$.

Step 3. We prove that the convex body K has constant width. By a characterization of convex body of constant width,[6] this is equivalent to show that for each pair H_1, H_2 of parallel supporting planes of K there exist $p \in H_1 \cap \partial K$ and $q \in H_2 \cap \partial K$ with $p \neq q$ such that the chord $[p, q] := \{tp + (1-t)\,q \mid 0 \leqslant t \leqslant 1\}$ is orthogonal to H_1, H_2, i.e., such that $(p-q)/|p-q|$ is a common normal vector of H_1 and H_2. Let H_1, H_2 be two parallel supporting planes of K. Then there exist $\omega \in \mathbf{S}^{n-1}$, $p \in H_1 \cap \partial K$ and $q \in H_2 \cap \partial K$ with $\langle p, \omega\rangle \neq \langle q, \omega\rangle$ such that $H_1 = \{v \in \mathbf{R}^n \mid \langle v-p, \omega\rangle = 0\}$, $H_2 = \{v \in \mathbf{R}^n \mid \langle v-q, \omega\rangle = 0\}$. We may suppose that $\langle q, \omega\rangle < \langle p, \omega\rangle$. This implies that

$$\langle v-p, \omega\rangle \leqslant 0 \qquad \text{and} \qquad \langle v-q, \omega\rangle \geqslant 0 \qquad \forall v \in K. \tag{4.4}$$

Since $p, q \in K$, by Eq. (4.1) we may assume that $\frac{1}{2}(p+q) + \frac{1}{2}|p-q|\,\omega \in K$. Then using the first inequality in (4.4) we have $\langle \frac{1}{2}(p+q) + \frac{1}{2}|p-q|\,\omega - p, \omega\rangle \leqslant 0$ which implies that $|p-q| \leqslant \langle p-q, \omega\rangle$. Thus $(p-q)/|p-q| = \omega$. Similarly, if $\frac{1}{2}(p+q) - \frac{1}{2}|p-q|\,\omega \in K$, then using the second inequality in (4.4) we still obtain $(p-q)/|p-q| = \omega$. Therefore K has constant width.

Step 4. Suppose $n \geqslant 3$. We now prove that K is a ball. After a translation we can assume that $0 \in K^\circ$. In this case, by Theorem MSW (with $p_0 = 0$), we need only to show that for any $n-1$-dimensional subspace $\Pi = \{v \in \mathbf{R}^n \mid \langle v, e_0\rangle = 0\} (e_0 \in \mathbf{S}^{n-1})$, the section $\Pi \cap K$ is an $n-1$-dimensional convex body of constant width. Let $\{e_0, e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $\mathbf{R}^n$. Define $L : \Pi \to \mathbf{R}^{n-1}$ by $L(v) = x = (x_1, x_2, \dots, x_{n-1})$ for $v = \sum_{k=1}^{n-1} x_k e_k \in \Pi$. Then L is a linear isometry between Π and $\mathbf{R}^{n-1}$, and since $0 \in K^\circ$, the set $K_1 := L(\Pi \cap K)$ is an $n-1$-dimensional convex body in $\mathbf{R}^{n-1}$ with $n-1 \geqslant 2$. Thus by the above result we need only to prove that the set K_1 satisfies Eq. (4.1) of $n-1$-dimensional case. For any $x, y \in K_1$ and any $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}) \in \mathbf{S}^{n-2}$, let $v = L^{-1}(x)$, $v_* = L^{-1}(y)$ and $\omega = \sum_{k=1}^{n-1} \sigma_k e_k$. Then $v, v_* \in \Pi \cap K$, $\omega \in \Pi \cap \mathbf{S}^{n-1}$ and

$$\frac{x+y}{2} \pm \frac{|x-y|}{2}\,\sigma = L\left(\frac{v+v_*}{2} \pm \frac{|v-v_*|}{2}\,\omega\right).$$

By Eq. (4.1) for K we see that either $\frac{1}{2}(x+y) + \frac{1}{2}|x-y|\,\sigma \in K_1$ or $\frac{1}{2}(x+y) - \frac{1}{2}|x-y|\,\sigma \in K_1$. Thus K_1 also satisfies Eq. (4.1) and therefore K_1

and, equivalently, $\Pi \cap K$ is an $n-1$-dimensional convex body of constant width. ▌

Now we give the classification of equilibria of Eq. (BFD).

Theorem 3. The equation (1.6) with (1.7) has only two classes of solutions: The first ones, corresponding to $S(f) > 0$, are Fermi–Dirac distributions:

$$f(v) = F_{a,b}(v) := \frac{ae^{-b|v-v_0|^2}}{1+\varepsilon ae^{-b|v-v_0|^2}} \qquad \text{a.e.} \quad v \in \mathbf{R}^3 \tag{4.5}$$

with constants $a > 0$, $b > 0$ and $v_0 \in \mathbf{R}^3$. The second ones, corresponding to $S(f) = 0$, are characteristic functions of balls (multiplying $1/\varepsilon$):

$$f(v) = \frac{1}{\varepsilon} 1_{\{|v-v_0| \leqslant R\}}, \qquad \text{a.e.} \quad v \in \mathbf{R}^3. \tag{4.6}$$

Proof. Suppose $\varepsilon = 1$. Let f be a solution of (1.6)–(1.7). In the following we denote for real function φ and constants c, c_1, c_2, $\mathbf{R}^3(\varphi > c) = \{v \in \mathbf{R}^3 \mid \varphi(v) > c\}$, $\mathbf{R}^3(c_1 < \varphi < c_2) = \{v \in \mathbf{R}^3 \mid c_1 < \varphi(v) < c_2\}$, etc.

Case 1: *S(f)* > 0. By our definition of $S(f)$, this is equivalent to $\text{mes}(\mathbf{R}^3(0 < f < 1)) > 0$. We now prove that in this case f is a Fermi–Dirac distribution. Let $w(t) = t^3(1-t^2)^{3/2}$ $(0 \leqslant t \leqslant 1)$, $W(z, \omega) = w(|z|^{-1}|\langle z, \omega \rangle|)$. Consider two functions

$$\mathscr{I}_f(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} W(v - v_*, \omega)\, f(v')\, f(v'_*)(1 - f(v_*))\, d\omega\, dv_*,$$

$$\mathscr{J}_f(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} W(v - v_*, \omega)\, f(v_*)(1 - f(v'))(1 - f(v'_*))\, d\omega\, dv_*$$

Multiplying $W(v - v_*, \omega)$ to both sides of equation (1.6) and then taking integration with respect to (v_*, ω) we have, for a null set $Z \subset \mathbf{R}^3$,

$$f(v)[\mathscr{I}_f(v) + \mathscr{J}_f(v)] = \mathscr{I}_f(v), \qquad v \in \mathbf{R}^3 \backslash Z. \tag{4.7}$$

The functions $\mathscr{I}_f$, $\mathscr{J}_f$ possess the following properties:

(a) If $g \in L^1(\mathbf{R}^3)$ and $0 \leqslant g \leqslant 1$, then

$$|\mathscr{I}_f(v) - \mathscr{I}_g(v)|,\ |\mathscr{J}_f(v) - \mathscr{J}_g(v)| \leqslant 12\pi\, \|f - g\|_{L^1} \qquad \forall v \in \mathbf{R}^3. \tag{4.8}$$

In particular, if $f = g$ a.e. on $\mathbf{R}^3$, then $\mathscr{I}_f \equiv \mathscr{I}_g$, $\mathscr{J}_f \equiv \mathscr{J}_g$ on $\mathbf{R}^3$.

In fact using Lemma 1 with $\Psi(r) \equiv 1$ we have

$$|\mathscr{I}_f(v) - \mathscr{I}_g(v)|,\ |\mathscr{J}_f(v) - \mathscr{J}_g(v)|$$

$$\leqslant \iint_{\mathbf{R}^3 \times \mathbf{S}^2} W(v - v_*, \omega)[|(f-g)(v')|$$

$$+ |(f-g)(v'_*)| + |(f-g)(v_*)|]\, d\omega\, dv_*$$

$$\leqslant 12\pi\, \|f - g\|_{L^1}, \qquad \forall v \in \mathbf{R}^3.$$

(b) $\mathscr{I}_f$, $\mathscr{J}_f$ are continuous on $\mathbf{R}^3$: Denote $f_h(v) = f(v+h)$. We have

$$|\mathscr{I}_f(v+h) - \mathscr{I}_f(v)|,\ |\mathscr{J}_f(v+h) - \mathscr{J}_f(v)| \leqslant 12\pi\, \|f_h - f\|_{L^1} \quad \forall v, h \in \mathbf{R}^3. \tag{4.9}$$

In fact we have $\mathscr{I}_f(v+h) = \mathscr{I}_{f_h}(v)$, $\mathscr{J}_f(v+h) = \mathscr{J}_{f_h}(v)$, so (4.9) follows from (4.8).

(c) The set $\mathbf{R}^3(\mathscr{I}_f > 0) \cap \mathbf{R}^3(\mathscr{J}_f > 0)$ is non-empty.

In fact, since $\text{mes}(\mathbf{R}^3(0 < f < 1)) > 0$, there is a Lebesgue point v of f satisfying $0 < f(v) < 1$. Let $B_r(v)$ denote an open ball with center v and radius $r > 0$, and let

$$L_v(r) = \frac{1}{\text{mes}(B_r)} \int_{B_r(v)} |f(v_*) - f(v)|\, dv_*.$$

In Lemma 1, choose $\Psi(r) = 1_{\{0 \leqslant r < \delta\}}$ for $\delta > 0$. Let $A = 4\pi \int_0^{\pi/2} \sin(\theta)\, w(\cos\theta)\, d\theta$. Then by Lemma 1 we have

$$\left| \frac{1}{\text{mes}(B_\delta)} \iint_{B_\delta(v) \times \mathbf{S}^2} W(v - v_*, \omega)\, f' f'_* (1 - f_*)\, d\omega\, dv_* - A[f(v)]^2 (1 - f(v)) \right|$$

$$\leqslant \frac{1}{\text{mes}(B_\delta)} \iint_{\mathbf{R}^3 \times \mathbf{S}^2} W(v - v_*, \omega)\, 1_{\{|v_* - v| < \delta\}}$$

$$\times [|f(v') - f(v)| + |f(v'_*) - f(v)| + |f(v_*) - f(v)|]\, d\omega\, dv_*$$

$$\leqslant 4\pi \int_0^{\pi/2} \sin(\theta)\, w(\cos\theta)[L_v(\delta\cos\theta) + L_v(\delta\sin\theta) + L_v(\delta)]\, d\theta$$

$$\to 0 \qquad (\delta \to 0)$$

since $L_v(r) \to 0 (r \to 0)$. Thus for sufficiently small $\delta > 0$,

$$\mathscr{I}_f(v) \geqslant \iint_{B_\delta(v)\times \mathbf{S}^2} W(v-v_*, \omega)\, f' f'_*(1-f_*)\, d\omega\, dv_*$$
$$> \tfrac{1}{2}\,\mathrm{mes}(B_\delta)\, A[f(v)]^2\,(1-f(v)) > 0.$$

Similarly, $\mathscr{J}_f(v) > 0$.

Now we define $g(v) = \mathscr{I}_f(v)/[\mathscr{I}_f(v) + \mathscr{J}_f(v)]$ if $\mathscr{I}_f(v) + \mathscr{J}_f(v) > 0$; $g(v) = f(v)$ if $\mathscr{I}_f(v) + \mathscr{J}_f(v) = 0$. Then by (4.7), $g = f$ a.e. on $\mathbf{R}^3$ and therefore by property (a), $\mathscr{I}_f \equiv \mathscr{I}_g$, $\mathscr{J}_f \equiv \mathscr{J}_g$. We need to prove that $\mathcal{O} := \mathbf{R}^3(\mathscr{I}_g > 0) \cap \mathbf{R}^3(\mathscr{J}_g > 0) = \mathbf{R}^3$. Since properties (b), (c) imply that $\mathcal{O}$ is open and nonempty, we may suppose that for some $\delta > 0$, $B_\delta(0) \subset \mathcal{O}$. Let $\lambda = \frac{1}{2}(1+\sqrt{3/2}\,)$, $\eta = \frac{1}{2}(\sqrt{3/2} - 1)\,\delta$, and

$$\mathcal{O}_\delta(v) = \{(v_*, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2 \mid |v_*| < \eta,\ v_* \neq v,\ \sqrt{1/3} < \cos(\theta) < \sqrt{2/3}\}$$

where $\theta = \arccos(|\langle v - v_*, \omega\rangle|/|v-v_*|)$. By the elementary inequalities

$$|v'| \leqslant \sin(\theta)\,|v| + \cos(\theta)\,|v_*|, \qquad |v'_*| \leqslant \cos(\theta)\,|v| + \sin(\theta)\,|v_*|$$

we see that if $v \in B_{\lambda\delta}(0)$ then $v_*, v', v'_* \in B_\delta(0)$ for all $(v_*, \omega) \in \mathcal{O}_\delta(v)$. Since $0 < g = \mathscr{I}_g/(\mathscr{I}_g + \mathscr{J}_g) < 1$ on $B_\delta(0) \subset \mathcal{O}$, this implies that $g(v')\,g(v'_*)(1 - g(v_*)) > 0$, $g(v_*)(1-g(v'))(1-g(v'_*)) > 0$ for all $(v_*, \omega) \in \mathcal{O}_\delta(v)$. Therefore by definition of $\mathscr{I}_g$ and $\mathscr{J}_g$ we have $\mathscr{I}_g(v) > 0$, $\mathscr{J}_g(v) > 0$ for all $v \in B_{\lambda\delta}(0)$. Here we have used an obvious fact that the sets $\mathcal{O}_\delta(v)$ have positive measure with respect to the measure $d\omega\, dv_*$. Thus $B_{\lambda\delta}(0) \subset \mathcal{O}$. Iteratively, we obtain $B_{\lambda^n\delta}(0) \subset \mathcal{O}$, $n = 1, 2,\ldots$, and so $\mathcal{O} = \mathbf{R}^3$. Therefore $0 < g(v) < 1$ for all $v \in \mathbf{R}^3$ and g is continuous on $\mathbf{R}^3$. Since $g = f$ a.e. on $\mathbf{R}^3$, it follows that g satisfies Eq. (1.6) (with $\varepsilon = 1$). Thus $(\frac{g}{1-g})'(\frac{g}{1-g})'_* = (\frac{g}{1-g})(\frac{g}{1-g})_*$ on $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$, and so by a well known result of Arkeryd[1, 5, 20] we conclude $g(v) = ae^{-b|v-v_0|^2}/(1 + ae^{-b|v-v_0|^2})$ for some constants $a > 0$, $b > 0$ and $v_0 \in \mathbf{R}^3$.

Case 2: $S(f) = 0$. This is equivalent to $\mathrm{mes}(\mathbf{R}^3(0 < f < 1)) = 0$. In this case we prove that f is a characteristic function of a ball. Let $E = \mathbf{R}^3(f = 1)$. Since $0 \leqslant f \leqslant 1$, we have $f(v) = 1_E(v)$ a.e. $v \in \mathbf{R}^3$. And in the following we can assume that E is a Borel set. Multiplying $1_E(v)$ to both sides of Eq. (1.6) (for $\varepsilon = 1$) leads to a single equation

$$1_E(v)\, 1_E(v_*)[1 - 1_E(v')][1 - 1_E(v'_*)] = 0 \qquad \text{a.e.} \quad (v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2. \tag{4.10}$$

Using integration on $\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2$ with suitable changes of variables we see that the equation (4.10) is equivalent to the equation (4.1) in 3-dimension case, i.e.,

$$1_E(v)\,1_E(v_*)\left[1-1_E\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\,\omega\right)\right]\left[1-1_E\left(\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\,\omega\right)\right]=0 \tag{4.11}$$

for a.e. $(v, v_*, \omega)\in\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2$. Our proof is divided into several steps.

Step 1. We first prove that the set E is essentially bounded, i.e., there exists a null set $Z_0\subset\mathbf{R}^3$ such that $E\setminus Z_0$ is a bounded set. For $0<\delta<1$, we compute (changing variable $r=\langle t\zeta-v, \omega\rangle$)

$$\begin{aligned}
&4\pi\,\mathrm{mes}(E)\\
&\geqslant\frac{1}{2}\int_{\mathbf{S}^2}d\zeta\int_{\mathbf{S}^2}1_{\{|\langle\zeta,\,\omega\rangle|\geqslant\delta\}}\,d\omega\int_{-\infty}^{\infty}r^2 1_E(v+r\omega)\,dr\\
&\geqslant\frac{1}{2}\int_{\mathbf{S}^2}d\zeta\int_{\mathbf{S}^2}1_{\{|\langle\zeta,\,\omega\rangle|\geqslant\delta\}}\,d\omega\int_{-\infty}^{\infty}\langle t\zeta-v, \omega\rangle^2\,|\langle\zeta, \omega\rangle|\\
&\quad\times 1_E(v+\langle t\zeta-v, \omega\rangle\,\omega)\,1_E(t\zeta)\,dt\\
&=\int_{\mathbf{S}^2}d\omega\int_{|\langle\frac{v_*}{|v_*|},\,\omega\rangle|\geqslant\delta}\frac{1}{|v_*|^2}\langle v_*-v, \omega\rangle^2\left|\left\langle\frac{v_*}{|v_*|}, \omega\right\rangle\right|1_E(v')\,1_E(v_*)\,dv_*\\
&\geqslant\delta\int_E\frac{|v-v_*|^2}{|v_*|^2}\left(\int_{\mathbf{S}^2}\left|\left\langle\frac{v-v_*}{|v-v_*|}, \omega\right\rangle\right|^2 1_E(v')\right.\\
&\quad\left.\times 1_{\{|\langle\frac{v_*}{|v_*|},\,\omega\rangle|\geqslant\delta\}}\,d\omega\right)dv_*,\qquad v\in\mathbf{R}^3.
\end{aligned}\tag{4.12}$$

On the other hand, for any $v, v_*\in\mathbf{R}^3$ with $v_*\neq v$, using equality (2.3) with $\varphi(\omega)=1_E(v'_*)$ and writing

$$v'=v_*+\left\langle v-v_*, \frac{\sigma-\langle\sigma, \omega\rangle\,\omega}{\sqrt{1-\langle\sigma, \omega\rangle^2}}\right\rangle\frac{\sigma-\langle\sigma, \omega\rangle\,\omega}{\sqrt{1-\langle\sigma, \omega\rangle^2}},\qquad \sigma=\frac{v-v_*}{|v-v_*|}$$

we have

$$\int_{\mathbf{S}^2}|\langle\sigma, \omega\rangle|^2\,1_E(v')\,d\omega=\int_{\mathbf{S}^2}|\langle\sigma, \omega\rangle|\sqrt{1-\langle\sigma, \omega\rangle^2}\,1_E(v'_*)\,d\omega. \tag{4.13}$$

Moreover by Eq. (4.10) and Fubini's theorem, there is a null set $Z_0 \subset \mathbf{R}^3$ such that for any $v \in E \setminus Z_0$ there is a null set $Z_{0,v}$ such that for any $v_* \in E \setminus Z_{0,v}$, we have $(1 - 1_E(v'))(1 - 1_E(v'_*)) = 0$ a.e. $\omega \in \mathbf{S}^2$. Thus by (4.13) we obtain for any $v \in E \setminus Z_0$ and any $v_* \in E \setminus Z_{0,v}$

$$
\begin{aligned}
&\int_{\mathbf{S}^2} |\langle \sigma, \omega \rangle|^2 \, 1_E(v') \, d\omega \\
&\quad = \frac{1}{2} \int_{\mathbf{S}^2} |\langle \sigma, \omega \rangle| \, [|\langle \sigma, \omega \rangle| \, 1_E(v') + \sqrt{1 - \langle \sigma, \omega \rangle^2} \, 1_E(v'_*)] \, d\omega \\
&\quad \geqslant \frac{1}{2} \int_{\mathbf{S}^2} |\langle \sigma, \omega \rangle| \min\{|\langle \sigma, \omega \rangle|, \sqrt{1 - \langle \sigma, \omega \rangle^2}\} \, d\omega \\
&\quad = \frac{4\pi}{3} 2^{-3/2}
\end{aligned}
\tag{4.14}
$$

with $\sigma = (v - v_*)/|v - v_*|$. This gives

$$
\frac{4\pi}{3} 2^{-3/2} \leqslant \int_{\mathbf{S}^2} \left| \left\langle \frac{v - v_*}{|v - v_*|}, \omega \right\rangle \right|^2 1_E(v') \, 1_{\{|\langle \frac{v_*}{|v_*|}, \omega \rangle| \geqslant \delta\}} \, d\omega + 4\pi\delta.
$$

Choose $\delta = 3^{-1} 2^{-5/2}$. We obtain by (4.12) that

$$
4\pi \operatorname{mes}(E) \geqslant \frac{4\pi}{288} \int_E \frac{|v - v_*|^2}{|v_*|^2} \, dv_*, \qquad \forall v \in E \setminus Z_0. \tag{4.15}
$$

Since $|v - v_*|^2 \geqslant \frac{1}{2} |v|^2 - |v_*|^2$ and $0 < \operatorname{mes}(E) < \infty$, (4.15) implies that the set $E \setminus Z_0$ is bounded. Let Z_1 be a null set such that every $v \in E \setminus (Z_0 \cup Z_1)$ is a density point of $E \setminus Z_0$, i.e., v satisfies $\operatorname{mes}((E \setminus Z_0) \cap B_r(v))/\operatorname{mes}(B_r(v)) \to 1$ as $r \to 0$. Applying Fubini's theorem it is easily seen that the set $E \setminus (Z_0 \cup Z_1)$ also satisfies the Eq. (4.10) and Eq. (4.11). These properties allow us to assuming without loss generality that the set E is bounded and satisfies that every point $v \in E$ is a density point of E.

Step 2. Let $K = \bar{E}$ be the closure of E. Then K is compact and $\operatorname{mes}(K) > 0$. Since $\mathbf{R}^3 \setminus K$ is open, it is easily verified that the set K satisfies the Eq. (4.1) for all $(v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$. Thus by Proposition 2, K is a ball.

In the following two steps we prove that $\operatorname{mes}(K \setminus E) = 0$. Before doing these we need two equalities: Applying Fubini's theorem to Eq. (4.10) and Eq. (4.11) we have

$$1_E(v+r\sigma)[1-1_E(v+r\langle\sigma,\omega\rangle\,\omega)][1-1_E(v+r\sigma-r\langle\sigma,\omega\rangle\,\omega)]$$
$$=0 \qquad \text{a.e.} \quad (\sigma,\omega)\in \mathbf{S}^2\times\mathbf{S}^2 \tag{4.16}$$

for all $v\in E\setminus Z$ and all $r\in[0,\infty)\setminus Z_v^{(+)}$; and

$$\left[1-1_E\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\,\omega\right)\right]\left[1-1_E\left(\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\,\omega\right)\right]$$
$$=0 \qquad \text{a.e.} \quad \omega\in\mathbf{S}^2 \tag{4.17}$$

for all $(v,v_*)\in(E\times E)\setminus\mathscr{Z}$. Here Z is a null set in $\mathbf{R}^3$, $Z_v^{(+)}$ are null sets in $[0,\infty)$ that depend on v, and $\mathscr{Z}$ is a null set in $\mathbf{R}^3\times\mathbf{R}^3$.

Step 3. We will prove that for any $v_0\in K^\circ(=K\setminus\partial K)$ and any $R>0$ satisfying $B_R(v_0)\subset K^\circ$,

$$\operatorname{mes}(E\cap B_R(v_0))\geqslant 2^{-5/2}\operatorname{mes}(B_R(v_0)). \tag{4.18}$$

First of all, since $K=\bar{E}$ and since every point in E is a density point of E, it is easily seen that for any $z\in K$ and any $r>0$ we have $\operatorname{mes}(E\cap B_r(z))>0$. Now take a fixed $\omega_0\in\mathbf{S}^2$. For any small $0<\delta<\frac{1}{3}R$, let $a=v_0+(R-2\delta)\,\omega_0$, $b=v_0-(R-2\delta)\,\omega_0$, and let $E_a=E\cap B_\delta(a)$, $E_b=E\cap B_\delta(b)$. Since $a,b\in K$, we have $\operatorname{mes}(E_a)>0$, $\operatorname{mes}(E_b)>0$. Thus, as an exersise of measure theory, the set $\frac{1}{2}(E_a+E_b):=\{\frac{1}{2}(v+v_*)\mid v\in E_a,\, v_*\in E_b\}$ contains a ball. Since $\frac{1}{2}(E_a+E_b)\subset K$, this implies that $\operatorname{mes}(E\cap[\frac{1}{2}(E_a+E_b)])>0$. Now we need to prove that

$$I:=\int_E\left(\int_{\mathbf{R}^3}1_{E_a}(x+y)\,1_{E_b}(x-y)\,dy\right)dx>0.$$

Let $I(x)$ be the inner integration with respect to y, and take any $x\in E\cap[\frac{1}{2}(E_a+E_b)]$. We have $x=\frac{1}{2}(a_x+b_x)$ for some $a_x\in E_a$, $b_x\in E_b$. Since for sufficiently small $r>0$, $B_r(a_x)\subset B_\delta(a)$, $B_r(b_x)\subset B_\delta(b)$, and a_x, b_x are density points of E, it follows that

$$\frac{1}{\operatorname{mes}(B_r)}I(x)\geqslant\frac{1}{\operatorname{mes}(B_r)}\int_{B_r(0)}1_{E_a}(a_x+z)\,1_{E_b}(b_x-z)\,dz$$
$$\geqslant\frac{1}{\operatorname{mes}(B_r)}\left[\int_{B_r(0)}1_E(a_x+z)\,dz+\int_{B_r(0)}1_E(b_x-z)\,dz\right]-1\to 1$$

when $r\to 0$. Thus $I(x)>0$ for all $x\in E\cap[\frac{1}{2}(E_a+E_b)]$ and therefore $I>0$.

Recalling that the sets Z and $\mathscr{L}$ are null sets in $\mathbf{R}^3$ and in $\mathbf{R}^3 \times \mathbf{R}^3$ respectively, the positivity of I implies that

$$\int_{E\setminus Z} \left(\int_{\mathbf{R}^3} 1_{E_a}(x+y)\, 1_{E_b}(x-y)\, 1_{(E\times E)\setminus \mathscr{L}}(x+y, x-y)\, dy \right) dx > 0.$$

Thus there is $c \in E\setminus Z$ such that for a null set $Z_c^{(+)} \subset [0, \infty)$

$$\int_{\mathbf{R}^3} 1_{E_a}(c+y)\, 1_{E_b}(c-y)\, 1_{(E\times E)\setminus \mathscr{L}}(c+y, c-y)\, 1_{[0,\infty)\setminus Z_c^{(+)}}(|y|)\, dy > 0.$$

Thus there is $y_1 \in \mathbf{R}^3$ which together with c has the following properties:

$$\text{(i)}\quad c\in E\setminus Z; \qquad \text{(ii)}\quad c+y_1 \in E_a,\ c-y_1\in E_b;$$

$$\text{(iii)}\quad (c+y_1, c-y_1) \in (E\times E)\setminus \mathscr{L}; \qquad \text{(iv)}\quad R_1 := |y_1| \in [0,\infty)\setminus Z_c^{(+)}.$$

By the "a.e." conditions on Eqs. (4.16) and (4.17), these properties give the following inequalities:

$$1_E(c+R_1\langle\sigma,\omega\rangle\,\omega) + 1_E(c+R_1\sigma - R_1\langle\sigma,\omega\rangle\,\omega) \geqslant 1_E(c+R_1\sigma) \tag{4.19}$$

for a.e. $(\sigma, \omega) \in \mathbf{S}^2 \times \mathbf{S}^2$, and

$$1_E(c+R_1\omega) + 1_E(c-R_1\omega) \geqslant 1 \qquad \text{a.e.} \quad \omega \in \mathbf{S}^2. \tag{4.20}$$

Also, by $|a-b| = 2(R-2\delta)$, $R_1 = \frac{1}{2}|c+y_1-(c-y_1)|$, and $v_0 = \frac{1}{2}(a+b)$, we have $R-3\delta \leqslant R_1 \leqslant R-\delta$ and $|c-v_0| \leqslant \frac{1}{2}(|c+y_1-a| + |c-y_1-b|) < \delta$. Thus $B_{R_1}(c) \subset B_R(v_0)$. Now let $\psi(t) = t\cdot \min\{t, \sqrt{1-t^2}\}$, $t\in[0,1]$. By the formula (4.13) (with $v=c$, $v_* = c+R_1\sigma$) we have

$$\begin{aligned}
&\int_{\mathbf{S}^2} |\langle\sigma,\omega\rangle|^2\, 1_E(c+R_1\langle\sigma,\omega\rangle\,\omega)\, d\omega \\
&\quad = \int_{\mathbf{S}^2} |\langle\sigma,\omega\rangle| \sqrt{1-\langle\sigma,\omega\rangle^2}\, 1_E(c+R_1\sigma - R_1\langle\sigma,\omega\rangle\,\omega)\, d\omega \\
&\quad \geqslant \frac{1}{2}\int_{\mathbf{S}^2} \psi(|\langle\sigma,\omega\rangle|)[1_E(c+R_1\langle\sigma,\omega\rangle\,\omega) + 1_E(c+R_1\sigma - R_1\langle\sigma,\omega\rangle\,\omega)]\, d\omega.
\end{aligned}$$

Thus by (4.19), (4.20) and (4.14) we obtain

$$\iint_{S^2\times S^2} |\langle\sigma,\omega\rangle|^2\, 1_E(c+R_1\langle\sigma,\omega\rangle\,\omega)\,d\omega\,d\sigma$$

$$\geqslant \frac{1}{2}\iint_{S^2\times S^2} \psi(|\langle\sigma,\omega\rangle|)\,1_E(c+R_1\sigma)\,d\omega\,d\sigma$$

$$=\frac{1}{4}\iint_{S^2\times S^2} \psi(|\langle\sigma,\omega\rangle|)[1_E(c+R_1\sigma)+1_E(c-R_1\sigma)]\,d\omega\,d\sigma$$

$$\geqslant \frac{1}{4}\iint_{S^2\times S^2} \psi(|\langle\sigma,\omega\rangle|)\,d\omega\,d\sigma=\frac{4\pi}{3}\cdot 2^{-5/2}\cdot 4\pi.$$

On the other hand, we compute

$$\iint_{S^2\times S^2} |\langle\sigma,\omega\rangle|^2\, 1_E(c+R_1\langle\sigma,\omega\rangle\,\omega)\,d\omega\,d\sigma$$

$$=\frac{4\pi}{R_1^3}\int_0^{R_1} r^2\int_{S^2} 1_E(c+r\omega)\,d\omega\,dr=\frac{4\pi}{R_1^3}\,\mathrm{mes}(E\cap B_{R_1}(c)).$$

Therefore $\mathrm{mes}(E\cap B_{R_1}(c))\geqslant \frac{4\pi}{3}R_1^3\cdot 2^{-5/2}$ and so $\mathrm{mes}(E\cap B_R(v_0))\geqslant \frac{4\pi}{3}(R-3\delta)^3\cdot 2^{-5/2}$. Letting $\delta\to 0$ leads to the inequality (4.18).

Step 4. We prove that $\mathrm{mes}(K\setminus E)=0$. This will complete the proof of the theorem. Since $K=\bar{E}$ is a ball, it needs only to show that the set $\tilde{Z}:=K^\circ\setminus E$ has measure zero. Suppose to the contrary that $\mathrm{mes}(\tilde{Z})>0$. Then there is a $v_0\in\tilde{Z}$ such that $\mathrm{mes}(\tilde{Z}\cap B_r(v_0))/\mathrm{mes}(B_r(v_0))\to 1$ as $r\to 0$. But the inequality (4.18) implies that for all small $r>0$ satisfying $B_r(v_0)\subset K^\circ$ we have $\mathrm{mes}(\tilde{Z}\cap B_r(v_0))\leqslant(1-2^{-5/2})\,\mathrm{mes}(B_r(v_0))$. This is a contradiction. Thus $\mathrm{mes}(\tilde{Z})=0$. ∎

5. TEMPERATURE INEQUALITY AND TREND TO EQUILIBRIUM

We begin by dealing with certain moment equations and inequalities.

Proposition 3. Let $M_0>0$, $M_2>0$, and $v_0\in\mathbf{R}^3$. Then: there exists a unique Fermi–Dirac distribution $F_{a,b}$ with coefficients $a>0$, $b>0$ and v_0, such that

$$\int_{\mathbf{R}^3} F_{a,b}(v)\,dv=M_0,\qquad \int_{\mathbf{R}^3} F_{a,b}(v)|v-v_0|^2\,dv=M_2 \tag{5.1}$$

if and only if M_0, M_2 satisfy

$$\frac{M_2}{(M_0)^{5/3}} > \frac{3}{5}\left(\frac{3\varepsilon}{4\pi}\right)^{2/3}.$$

Proof. Introduce functions (for $s \geqslant 0$)

$$I_s(t) = \int_0^\infty \frac{r^s}{1+te^{r^2}}\,dr, \qquad P(t) = I_4(t)[I_2(t)]^{-5/3}, \qquad t > 0.$$

By calculation, (5.1) is equivalent to the the following equation system for $a, b > 0$

$$\left(\frac{\varepsilon}{4\pi}\right)^{2/3} P\left(\frac{1}{\varepsilon\, a}\right) = \frac{M_2}{(M_0)^{5/3}}, \qquad b = \left(\frac{4\pi}{\varepsilon M_0} I_2\left(\frac{1}{\varepsilon a}\right)\right)^{2/3}. \tag{5.2}$$

Thus we need only to show that

$$\frac{d}{dt} P(t) > 0 \quad \forall t > 0; \qquad \lim_{t\to 0+} P(t) = \frac{3^{5/3}}{5}, \quad \lim_{t\to\infty} P(t) = \infty. \tag{5.3}$$

Differentiation under integral sign gives

$$-\frac{d}{dt} I_s(t) = J_s(t) := \int_0^\infty \frac{r^s e^{r^2}}{(1+te^{r^2})^2}\,dr, \qquad t > 0;$$

and integration by parts gives $I_2(t) = \frac{2t}{3} J_4(t)$, $\frac{5}{3} I_4(t) = \frac{2t}{3} J_6(t)$. Thus for a function $P_1(t) > 0$ we have

$$\frac{d}{dt} P(t) = P_1(t)\{J_2(t)\, J_6(t) - [J_4(t)]^2\}, \qquad t > 0$$

Applying Cauchy–Schwarz inequality we have $J_2(t)\, J_6(t) > [J_4(t)]^2$. This proves $\frac{d}{dt} P(t) > 0$ for all $t > 0$. To prove the first limit in (5.3), we write $t = e^{-\rho}$ for $\rho > 0$ and define

$$K_s(\rho) = \frac{s+3}{2} \int_0^\infty \frac{u^{\frac{s+1}{2}}}{1+e^{\rho(u-1)}}\,du. \tag{5.4}$$

Making change of integral variable $r = \sqrt{\rho u}$ in $I_s(t)$ for $t = e^{-\rho}$ we obtain

$$P(e^{-\rho}) = \frac{3^{5/3}}{5} \cdot \frac{K_2(\rho)}{[K_0(\rho)]^{5/3}}, \qquad \rho > 0.$$

By splitting $\int_0^\infty = \int_0^1 + \int_1^\infty$ for (5.4) and using dominated convergence theorem, we have

$$K_s(\rho) \to \frac{s+3}{2} \int_0^1 u^{\frac{s+1}{2}}\, du = 1 \qquad (\rho \to \infty).$$

This proves the first limit. The second limit in (5.3) is obvious. ∎

Lemma 4. Given constants $0 < p < q < \infty$. Let ϕ be measurable on $[0, \infty)$ with $0 \leqslant \phi \leqslant 1$ and $0 < \int_0^\infty r^{q-1}\phi(r)\, dr < \infty$. Then

$$\left(p \int_0^\infty r^{p-1}\phi(r)\, dr\right)^{1/p} \leqslant \left(q \int_0^\infty r^{q-1}\phi(r)\, dr\right)^{1/q} \tag{5.5}$$

and the equality sign holds if and only if there is a constant $0 < R < \infty$ such that $\phi = 1_{[0, R]}$ a.e. on $[0, \infty)$.

Remark. As a referee commented, this lemma is a generalization of a certain L^p-inequality. In fact if one takes $\phi(r) = \mu(\{x \in \Omega \mid g(x) > r\})$ where μ is a probability measure and g is a nonnegative function in $L^q(\Omega, d\mu)$, then this lemma is not other than the statement that the $L^p(\Omega, d\mu)$-norm of g is monotonously increasing in p. And also there, equality holds only if g is a constant, which means that $\mu(\{x \in \Omega \mid g(x) > r\})$ must be a step function as indicated in this lemma. For general case, i.e., if we do not assume that ϕ is non-increasing, the proof of the lemma will be different from this argument.

Proof of Lemma 4. Consider

$$\Phi(r) = \left(p \int_0^r t^{p-1}\phi(t)\, dt\right)^{q/p} - q \int_0^r t^{q-1}\phi(t)\, dt, \qquad r \geqslant 0.$$

By $0 \leqslant \phi(t) \leqslant 1$ and $q/p > 1$, we have

$$\frac{d}{dr}\Phi(r) = \left\{\frac{q}{p}\left(p \int_0^r t^{p-1}\phi(t)\, dt\right)^{(q/p)-1} p r^{p-1} - q r^{q-1}\right\} \phi(r) \leqslant 0 \tag{5.6}$$

for all $r \in [0, \infty) \setminus Z_0$. Here Z_0 is a null set. This gives (5.5) by the absolute continuity of Φ and $\Phi(0) = 0$. Now suppose that in (5.5) the equality sign holds. Then, since Φ is non-increasing, we have $\Phi(r) \equiv 0$ for all $r \geqslant 0$. Let $I = \{r \in (0, \infty) \setminus Z_0 \mid \phi(r) > 0\}$. Obviously I is non-empty. For any $r \in I$, the equality signs in (5.6) imply that $p \int_0^r t^{p-1}\phi(t)\, dt = r^p$. Since $0 \leqslant \phi \leqslant 1$, this implies that $\phi(t) = 1$ a.e. on $[0, r]$ $\forall r \in I$. Thus, by assumption, the number $R := \sup I$ must be finite and therefore $\phi = 1_{[0,R]}$ a.e. on $[0, \infty)$. ∎

Proposition 4. Let $f \in L^1_2(\mathbf{R}^3)$ satisfy $0 \leqslant f \leqslant 1/\varepsilon$ and $\int_{\mathbf{R}^3} f(v)\, dv > 0$. Let

$$M_0 = \int_{\mathbf{R}^3} f(v)\, dv, \quad M_2 = \int_{\mathbf{R}^3} f(v)\, |v - v_0|^2 dv, \quad v_0 = \frac{1}{M_0} \int_{\mathbf{R}^3} f(v)\, v\, dv. \tag{5.7}$$

Then

$$\frac{M_2}{(M_0)^{5/3}} \geqslant \frac{3}{5} \left(\frac{3\varepsilon}{4\pi} \right)^{2/3} \tag{5.8}$$

and the equality sign holds if and only if f is a second equilibrium (4.6).

Proof. Still suppose $\varepsilon = 1$. Let

$$\bar{f}(r) = \frac{1}{4\pi} \int_{\mathbf{S}^2} f(v_0 + r\omega)\, d\omega.$$

Then (5.8) is equivalent to the inequality

$$\left(5 \int_0^\infty r^4 \bar{f}(r)\, dr \right)^{1/5} \geqslant \left(3 \int_0^\infty r^2 \bar{f}(r)\, dr \right)^{1/3}$$

which does hold by Lemma 4. Also, since $0 \leqslant f \leqslant 1$ on $\mathbf{R}^3$, it is easily seen that the two equalities $\bar{f}(r) = 1_{\{0 \leqslant r \leqslant R\}}$ a.e. $r \in [0, \infty)$ and $f(v) = 1_{\{|v - v_0| \leqslant R\}}$ a.e. $v \in \mathbf{R}^3$ are equivalent. This proves the proposition. ∎

In the following the function f in (5.7) for defining M_0, M_2 and v_0 will be taken an initial datum f_0 of a conservative solution of Eq. (BFD). By conservation of the mass, momentum and energy, the temperature T of the gas (see ref. 7, Chapter 2; ref. 20, pp.43–44]) and the Fermi temperature T_F (see ref. 16, pp. 220–221 for ideal Fermi systems) can be written (with the Boltzmann's constant k_B)

$$T = \frac{m}{3\, k_B} \cdot \frac{M_2}{M_0}, \qquad T_F = \left(\frac{3M_0}{4\pi\, \mathrm{g}} \right)^{2/3} \cdot \frac{h^2}{2m\, k_B}.$$

Since $\varepsilon=(h/m)^3/g$, the inequality (5.8) is equivalent to the temperature inequality:

$$\frac{T}{T_F}=\frac{2}{3}\left(\frac{4\pi}{3\varepsilon}\right)^{2/3}\frac{M_2}{(M_0)^{5/3}}\geqslant\frac{2}{5}. \tag{5.9}$$

Theorem 4. Suppose the collision kernel B is given by (1.1)–(1.2). Let $f_0\in L^1_2(\mathbf{R}^3)$ satisfy $0\leqslant f_0\leqslant 1/\varepsilon$ and $\|f_0\|_{L^1_0}>0$. Let f be a conservative solution of Eq. (BFD) with $f|_{t=0}=f_0$. Then we have:

(1) The temperature inequality $T\geqslant\frac{2}{5}T_F$ holds.

(2) If $T=\frac{2}{5}T_F$, then f is a second equilibrium, i.e., for all $t\in[0,\infty)$ and for almost all $v\in\mathbf{R}^3$,

$$f(v,t)\equiv f_0(v)\equiv\frac{1}{\varepsilon}1_{\{|v-v_0|\leqslant R\}}.$$

(3) If $T>\frac{2}{5}T_F$, then for any sequence $\{t_n\}_{n=1}^{\infty}\subset[0,\infty)$ satisfying $\lim_{n\to\infty}t_n=\infty$, there exist a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ and a Fermi–Dirac distribution F, such that

$$f(\cdot,t_{n_k})\rightharpoonup F\quad(k\to\infty)\quad\text{weakly in } L^1(\mathbf{R}^3).$$

In particular, if f also satisfies that for some $t_0>0$,

$$\sup_{t\geqslant t_0}\int_{|v|>R}f(v,t)\,|v|^2\,dv\to 0\qquad(R\to\infty) \tag{5.10}$$

(for instance f is a solution obtained in Theorem 2 for hard potentials), then

$$f(\cdot,t)\rightharpoonup F_{a,b}\quad(t\to\infty)\quad\text{weakly in } L^1(\mathbf{R}^3)$$

where $F_{a,b}$ is the unique Fermi–Dirac distribution determined by the moment equation system (5.1) with $v_0=\frac{1}{M_0}\int_{\mathbf{R}^3}f_0(v)\,v\,dv$.

Proof. Part (1) has been shown above. Part (2) follows from Proposition 4 (the conclusion for equality sign) and the condition that f conserves the mass, mean velocity and energy. To prove Part (3), we assume $\varepsilon=1$. Suppose $t_n\geqslant 0$ and $\lim_{n\to\infty}t_n=\infty$. By weak compactness of $\{f(\cdot,t)\mid t\geqslant 0\}$, there exist a subsequence, still denote it by $\{t_n\}_{n=1}^{\infty}$, and a function $F\in L^1(\mathbf{R}^3)$, such that $f(\cdot,t_n)\rightharpoonup F$ $(n\to\infty)$ weakly in $L^1(\mathbf{R}^3)$. We first prove that F is an equilibrium. It is obvious that $F\in L^1_2(\mathbf{R}^3)$,

$\|F\|_{L_0^1} = \|f_0\|_{L_0^1} > 0$, and we can assume that $0 \leqslant F(v) \leqslant 1$ for all $v \in \mathbf{R}^3$. By Theorem 1, the entropy $t \mapsto S(f(t))$ is continuous, bounded and monotone non-decreasing on $[0, \infty)$. Thus there exist sequences $\{\delta_n\}_{n=1}^{\infty}$, $\{\tau_n\}_{n=1}^{\infty}$ satisfying $\delta_n > 0$, $\tau_n \in [t_n, t_n + \delta_n]$ such that (see, e.g., ref. 14) $e(f(\tau_n)) \leqslant \delta_n \to 0$ $(n \to \infty)$. Thus for a constant $C = C(A_0, \beta, \|f_0\|_{L_2^1})$ we have $\|f(\tau_n) - f(t_n)\|_{L_0^1} \leqslant C|\tau_n - t_n| \to 0$ $(n \to \infty)$. This implies that $f(\cdot, \tau_n)$ also converge weakly to F. Next, let

$$D(f(t)) = \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} B\,|f' f'_*(1-f)(1-f_*) - f f_*(1-f')(1-f'_*)|\, d\omega\, dv_*\, dv$$

and, in the following inequality

$$|a-b| \leqslant \sqrt{a+b}\,\sqrt{\Gamma(a, b)}, \qquad a, b \geqslant 0$$

choose

$$a = f' f'_*(1-f)(1-f_*), \qquad b = f f_*(1-f')(1-f'_*).$$

Then by Cauchy–Schwarz inequality we have for some constant $C = C(A_0, \beta, \|f_0\|_{L_2^1})$

$$D(f(\tau_n)) \leqslant C\sqrt{e(f(\tau_n))} \to 0 \qquad (n \to \infty).$$

Since $|Q(f(\tau_n))^\wedge(\xi)| \leqslant D(f(\tau_n))$, this implies by Lemma 3 that

$$Q(F)^\wedge(\xi) = \lim_{n \to \infty} Q(f(\tau_n))^\wedge(\xi) = 0, \qquad \forall \xi \in \mathbf{R}^3.$$

Thus $Q(F)(v) = 0$ a.e. $v \in \mathbf{R}^3$ and therefore F is a solution of Eq. (BFD) independent of t. By the entropy identity (1.4) we have $e(F) = 0$. Since the kernel $B(z, \omega) > 0$ a.e. on $\mathbf{R}^3 \times \mathbf{S}^2$, this implies that F is an equilibrium. To prove that F is a Fermi–Dirac distribution, we need to prove

$$S(f(t)) \leqslant S(F) \qquad \forall t \geqslant 0. \tag{5.11}$$

Let $F_k(v) = (1 - \frac{2}{k})\, F(v) + \frac{1}{k} e^{-|v|}$, $k \geqslant 3$. Applying the estimate (3.1) to $g = F_k$ and using dominated convergence theorem we have $\lim_{k \to \infty} S(F_k) = S(F)$. Next, let $\psi_k(v) = \log[(1 - F_k(v))/F_k(v)]$. Then $|\psi_k(v)| \leqslant (\log k)(1 + |v|)$ and

$$|\psi_k(v)[F(v) - F_k(v)]| \leqslant \frac{2\log k}{k}[F(v) + e^{-|v|}](1 + |v|).$$

Since $y \mapsto -(1-y)\log(1-y) - y\log y$ is concave on $[0, 1]$, it follows that

$$S(f(t_n)) \leqslant S(F_k) + \int_{\mathbf{R}^3} \psi_k(v)[f(v, t_n) - F_k(v)]\, dv.$$

Therefore, first letting $n \to \infty$ then letting $k \to \infty$, we obtain (5.11) by monotonicity of the entropy. Now we assert that $S(F) > 0$. Otherwise, $S(F) = 0$, then (5.11) implies $S(f(t)) \equiv 0$ on $[0, \infty)$. By entropy identity (1.4) we have $e(f(t)) = 0$ for a.e. $t \in [0, \infty)$. Thus for some $t_0 > 0$, $f(v, t_0)$ is an equilibrium satisfying $S(f(t_0)) = 0$ and so by Theorem 3, $f(v, t_0)$ is a second equilibrium. Since f is a conservative solution, this implies by Proposition 4 and (5.9) that $T = \frac{2}{5} T_F$ which contradicts the condition $T > \frac{2}{5} T_F$. This proves $S(F) > 0$ and therefore by Theorem 3, F is a Fermi–Dirac distribution. Finally suppose f satisfies the condition (5.10). To prove that $f(\cdot, t) \rightharpoonup F_{a,b} (t \to \infty)$ weakly in $L^1(\mathbf{R}^3)$, it needs only to prove that for any sequence $\{t_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} t_n = \infty$, if $f(\cdot, t_n) \rightharpoonup F (n \to \infty)$ weakly in $L^1(\mathbf{R}^3)$, then F must be the same Fermi–Dirac distribution $F_{a,b}$. But, we have shown that any such a weak limit F must be a Fermi–Dirac distribution, and the condition (5.10) ensures that the five moments of F are equal to those of f_0. Therefore we conclude $F \equiv F_{a,b}$. ∎

Remark. For the BFD model, Csiszár–Kullback inequalities[2, 8, 10] for the entropy $S(f)$ hold for conservative solutions f and the relevant Fermi–Dirac distributions $F_{a,b}$. For example with L^1-distance we have

$$\|f(t) - F_{a,b}\|_{L^1}^2 \leqslant 2\, \|f_0\|_{L^1} [S(F_{a,b}) - S(f(t))], \qquad t \geqslant 0.$$

[A simple proof of such inequalities is given by starting from the identity (for convex ψ)

$$|y-x| = 2 \int_0^1 [(1-\tau)\, \psi''(x+\tau(y-x))\, |y-x|^2]^{1/2} \times [(1-\tau)(\psi''(x+\tau(y-x)))^{-1}]^{1/2}\, d\tau.$$

Then, for the BFD model, take $\psi(x) = (1-x)\log(1-x) + x\log x$ $(0 < x < 1)$ and make use of Cauchy–Schwarz inequality and Taylor formula to obtain an elementary inequality

$$|y-x| \leqslant 2[\psi(y) - \psi(x) - \psi'(x)(y-x)]^{1/2}\, [x/3 + y/6]^{1/2}$$

for all $0 < x < 1$ and all $0 \leqslant y \leqslant 1$. Then choose $x = \varepsilon F_{a,b}(v)$, $y = \varepsilon f(v, t)$, etc.]

But strong convergence to equilibrium as that for the original Boltzmann equation seems a hard problem because, for instance at low temperatures $0 < T/T_F - 2/5 < < 1$, the different equilibria $F_{a,b}(v)$ and $\frac{1}{\varepsilon} 1_{\{|v-v_0| \leqslant R\}}$ can be very close in L^1-distance and thus the solution f with the same mass, momentum and energy as those of $F_{a,b}$ may be close (in some sense) to both $F_{a,b}$ and $1/\varepsilon$ in different large parts of velocities. In view of (relative) entropy methods, this may be a trouble case (see refs. 3, 17, 19 and references therein). To see the closeness of the different equilibria, let $M_0 = \frac{4\pi}{3\varepsilon} R^3$ be fixed, and let $F_{a,b}$ be the unique Fermi–Dirac distribution determined by the equation system (5.1) where $M_2 > 0$ is given through M_0 and $T/T_F(>2/5)$ (see (5.9)). By (5.2) and (5.9) we have $2 \cdot 3^{-5/3} P(1/(\varepsilon a)) = T/T_F$, and $a \to \infty$ if and only if $T/T_F \to 2/5$. Thus there is $\delta_0 > 0$ such that if $0 < T/T_F - 2/5 < \delta_0$ then $\varepsilon a > 3$. Let $\rho = \log(\varepsilon a)(>1)$. By (5.2) for b and (5.4) for $K_0(\rho)$ and changing variable $r = \sqrt{\rho u}$ in $I_2(e^{-\rho})$ we compute $b = R^{-2}[K_0(\rho)]^{2/3} \rho$. Then with the identity $|x-y| = y - x + 2(x-y)^+$ we obtain

$$\int_{\mathbf{R}^3} \left| F_{a,b}(v) - \frac{1}{\varepsilon} 1_{\{|v-v_0| \leqslant R\}} \right| dv = \frac{3M_0}{K_0(\rho)} \int_{[K_0(\rho)]^{2/3}}^{\infty} \frac{u^{1/2}}{1+e^{\rho(u-1)}} du. \tag{5.12}$$

The integral in the right-hand side of (5.12) is not greater than $|K_0(\rho) - 1| + \int_1^\infty$ which tends to zero as $\rho \to \infty$ since $K_0(\rho) \to 1$ $(\rho \to \infty)$. Thus

$$\int_{\mathbf{R}^3} \left| F_{a,b}(v) - \frac{1}{\varepsilon} 1_{\{|v-v_0| \leqslant R\}} \right| dv \to 0 \qquad \text{when} \quad \frac{T}{T_F} \to \frac{2}{5}.$$

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